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***Research Article***

# ITERATION PRODUCT OF NATARAJAN METHODS OF SUMMABILITY

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**ABSTRACT**
Throughout the present paper, entries of infinite matrices and sequences are real or complex numbers. We consider the iteration product of Natarajan methods and prove a few inclusion theorems on this iteration product.

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***Key Words:*** *Regular Matrix, Natarajan (or (M, $\lambda_n$)) Method, Iteration Product, Inclusion Theorem*

## INTRODUCTION

Throughout the present paper, entries of infinite matrices and sequences are real or complex numbers. To make the paper self-contained, we recall the following. Given an infinite matrix $A \equiv (a_{nk})$, n, k = 0, 1, 2... and a sequence $x = \{x_k\}$, k = 0, 1, 2..., by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...,$$

where we suppose that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is summable A or A-summable to $\ell$. If $\lim_{n\to\infty} (Ax)_n = \ell$, whenever $\lim_{k\to\infty} x_k = m$, we say that A is conservative. Further, if $\ell = m$, A is said to be regular. The following result, which gives a characterization of a conservative and a regular matrix in terms of its entries, is well-known (Hardy, 1949).

**Theorem 1.1:** $A \equiv (a_{nk})$ is conservative if and only if

(i) $\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = \delta_k, \quad k = 0,1,2,...;$

and

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \delta.$

Further, A is regular if and only if (i), (ii), (iii) hold with $\delta_k = 0$, k = 0, 1, 2... and $\delta = 1$.

The (M, $\lambda_n$) method was introduced by Natarajan in 2013(a) and some of its nice properties were studied earlier by Natarajan in 2012, 2013(a), 2013(b), No date.

**Definition 1.2:** Let $\{\lambda_n\}$ be such that $\sum_{n=0}^{\infty} |\lambda_n| < \infty$. Then the (M, $\lambda_n$) method is defined by the infinite matrix $A \equiv (a_{nk})$, where

$$a_{nk} = \begin{cases} \lambda_{n-k}, & k \leq n; \\ 0, & k > n. \end{cases}$$

**Remark 1.3:** In this context, it is worthwhile to note that the $(M, \lambda_n)$ method reduces to the Y-method when $\lambda_0 = \lambda_1 = \dfrac{1}{2}$ and $\lambda_n = 0$, $n \geq 2$.

**Remark 1.4:** The $(M, \lambda_n)$ method is always conservative using the fact that $\sum_{n=0}^{\infty} |\lambda_n| < \infty$ and Theorem 1.1.

The following result is known (Natarajan, 2013(a), Theorem 2.3).

**Theorem 1.5:** $(M, \lambda_n)$ is regular if and only if

$$\sum_{n=0}^{\infty} \lambda_n = 1.$$

***Iteration Product of Natarajan Methods***

The following definition is needed in the sequel.

**Definition 2.1:** Given the infinite matrix methods A, B, we say that A is included in B (or B includes A), written as $A \subseteq B$, if whenever $x = \{x_k\}$ is A-summable to $\ell$, it is also B-summable to $\ell$.

The following result is extremely useful (see Hardy, 1949, p. 234, Theorem 176).

**Theorem 2.2:** If $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} |b_n| < \infty$, then $\lim_{n\to\infty} c_n = 0$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2, \ldots$.

For convenience, we denote the methods $(M, p_n)$, $(M, q_n)$ and $(M, t_n)$ by (M, p), (M, q) and (M, t) respectively.

In this section, we prove a few theorems on the iteration product of Natarajan methods.

**Theorem 2.3:** Let (M, p), (M, t) be regular methods. Then (M, t) (M, p) is also regular, where we define, for $x = \{x_k\}$,

((M, t) (M, p))(x) = (M, t) ((M, p)(x)).

**Proof:** Let $\{\alpha_n\}$ be the (M, p)-transform of $x = \{x_k\}$ and $\{\beta_n\}$ be the (M, t) (M, p)-transform of $x = \{x_k\}$ so that

$$\alpha_n = \sum_{k=0}^{n} p_k x_{n-k},$$

$$\beta_n = \sum_{k=0}^{n} t_k \alpha_{n-k}, \quad n = 0,1,2,\ldots.$$

Now,

$$\begin{aligned}
\beta_n &= \sum_{k=0}^{n} t_k \alpha_{n-k} \\
&= t_0\alpha_n + t_1\alpha_{n-1} + \cdots + t_{n-1}\alpha_1 + t_n\alpha_0 \\
&= t_0\left[\sum_{k=0}^{n} p_k x_{n-k}\right] + t_1\left[\sum_{k=0}^{n-1} p_k x_{n-1-k}\right] + \cdots + t_{n-1}[p_0x_1 + p_1x_0] + t_n[p_0x_0] \\
&= t_0[p_0x_n + p_1x_{n-1} + \cdots + p_nx_0] + t_1[p_0x_{n-1} + p_1x_{n-2} + \cdots + p_{n-1}x_0] \\
&\qquad + \cdots + t_{n-1}[p_0x_1 + p_1x_0] + t_n[p_0x_0] \\
&= (p_0t_0)x_n + (p_0t_1 + p_1t_0)x_{n-1} + \cdots + (p_0t_n + p_1t_{n-1} + \cdots + p_nt_0)x_0 \\
&= \sum_{k=0}^{n} c_{nk}x_k,
\end{aligned}$$

where

$$c_{nk} = \begin{cases} \sum_{\gamma=0}^{k} p_\gamma t_{k-\gamma}, & k \le n; \\ 0, & k > n. \end{cases}$$

Now,

$$\begin{aligned}
\sum_{k=0}^{\infty} |c_{nk}| &= \sum_{k=0}^{n} |c_{nk}| \\
&= \sum_{k=0}^{n} \left| \sum_{\gamma=0}^{k} p_\gamma t_{k-\gamma} \right| \\
&\le \sum_{k=0}^{n} \left( \sum_{\gamma=0}^{k} |p_\gamma| \, |t_{k-\gamma}| \right) \\
&= |p_0| \, |t_0| + \left( \sum_{\gamma=0}^{1} |p_\gamma| \, |t_{1-\gamma}| \right) + \left( \sum_{\gamma=0}^{2} |p_\gamma| \, |t_{2-\gamma}| \right) + \cdots + \left( \sum_{\gamma=0}^{n} |p_\gamma| \, |t_{n-\gamma}| \right) \\
&= |p_0| \, |t_0| + \left( |p_0| \, |t_1| + |p_1| \, |t_0| \right) + \left( |p_0| \, |t_2| + |p_1| \, |t_1| + |p_2| \, |t_0| \right) + \cdots \\
&\qquad + \left( |p_0| \, |t_n| + |p_1| \, |t_{n-1}| + \cdots + |p_n| \, |t_0| \right) \\
&= |p_0| \left( \sum_{k=0}^{n} |t_k| \right) + |p_1| \left( \sum_{k=0}^{n-1} |t_k| \right) + \cdots + |p_n| \left( |t_0| \right) \\
&\le \left( \sum_{k=0}^{\infty} |t_k| \right) \left( \sum_{k=0}^{\infty} |p_k| \right), \quad n = 0,1,2,...,
\end{aligned}$$

so that

$$\sup_{n \ge 0} \sum_{k=0}^{\infty} |c_{nk}| \le \left( \sum_{k=0}^{\infty} |t_k| \right) \left( \sum_{k=0}^{\infty} |p_k| \right) < \infty,$$

noting that $\sum_{k=0}^{\infty} |t_k| < \infty$ and $\sum_{k=0}^{\infty} |p_k| < \infty$. Using $\lim_{n\to\infty} p_n = 0$, $\sum_{n=0}^{\infty} |t_n| < \infty$ and Theorem 2.2, it follows that $\lim_{n\to\infty} c_{nk} = 0$, k = 0, 1, 2, ... . Now,

$$\begin{aligned}
\sum_{k=0}^{\infty} c_{nk} &= \sum_{k=0}^{n} c_{nk} \\
&= \sum_{k=0}^{n} \left( \sum_{\gamma=0}^{k} p_\gamma t_{k-\gamma} \right) \\
&= p_0 \left( \sum_{k=0}^{n} t_k \right) + p_1 \left( \sum_{k=0}^{n-1} t_k \right) + \cdots + p_n (t_0) \\
&= p_0 T_n + p_1 T_{n-1} + \cdots + p_n T_0, \text{ where } T_n = \sum_{k=0}^{n} t_k, \ n = 0,1,2,... \\
&= p_0 (T_n - 1) + p_1 (T_{n-1} - 1) + \cdots + p_n (T_0 - 1) + P_n, \text{ where } P_n = \sum_{k=0}^{n} p_k, \ n = 0,1,2,....
\end{aligned}$$

Since (M, t), (M, p) are regular, $\lim_{n\to\infty} T_n = \lim_{n\to\infty} P_n = 1,$ in view of Theorem 1.5. Using $\lim_{n\to\infty}(T_n - 1) = 0,$ $\sum_{k=0}^{\infty}|p_k| < \infty$ and Theorem 2.2, we have,

$$\lim_{n\to\infty}\left[p_0(T_n - 1) + p_1(T_{n-1} - 1) + \cdots + p_n(T_0 - 1)\right] = 0.$$

Thus,

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} c_{nk} = 0 + 1 = 1.$$

Consequently $(c_{nk})$ is regular in view of Theorem 1.1. In other words, (M, t) (M, p) is regular. This completes the proof of the theorem.

**Theorem 2.4:** Let (M, p), (M, q), (M, t) be regular methods. Then

(M, p) $\subseteq$ (M, q)

if and only if

(M, t) (M, p) $\subseteq$ (M, t) (M, q).

**Proof:** We write

$$r'_n = \sum_{k=0}^{n} p_k t_{n-k}, \quad r''_n = \sum_{k=0}^{n} q_k t_{n-k}, \quad n = 0,1,2,\ldots .$$

Let $r' = \{r'_n\},\ r'' = \{r''_n\},\ r'(x) = \sum_{n=0}^{\infty} r'_n x^n,\ r''(x) = \sum_{n=0}^{\infty} r''_n x^n,\ p(x) = \sum_{n=0}^{\infty} p_n x^n,\ q(x) = \sum_{n=0}^{\infty} q_n x^n$ and

$$t(x) = \sum_{n=0}^{\infty} t_n x^n.$$

Since (M, p), (M, q) and (M, t) are regular, (M, t) (M, p) and (M, t) (M, q) are regular too in view of Theorem 2.3. To prove the present theorem, it suffices to show that

$$\frac{r''(x)}{r'(x)} = \frac{q(x)}{p(x)}.$$

We first note that

$$r'(x) = p(x)t(x) \quad \text{and} \quad r''(x) = q(x)t(x)$$

so that

$$\frac{r''(x)}{r'(x)} = \frac{q(x)}{p(x)}.$$

We now use Theorem 3.1 of Natarajan (2013(a)) to arrive at the conclusion, thus completing the proof of the theorem.

In view of Theorem 3.1 of Natarajan (2013(a)), we can reformulate Theorem 2.4 as follows:

**Theorem 2.5:** For given regular methods (M, p), (M, q) and (M, t), the following statements are equivalent:

(i) (M, p) $\subseteq$ (M, q);

(ii) (M, t) (M, p) $\subseteq$ (M, t) (M, q);

and

(iii) $\sum_{n=0}^{\infty}|k_n| < \infty$ and $\sum_{n=0}^{\infty} k_n = 1,$

where $\frac{q(x)}{p(x)} = k(x) = \sum_{n=0}^{\infty} k_n x^n.$

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