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Research Article

CLASSICAL PLATE THEORY: ASYMPTOTIC APPROACH

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ABSTRACT
This paper deals with the study of mechanics of plate in some different way without altering the fundamental plate hypotheses and it is shown that our formulation agrees with Kirchoff-Love plate theory in classical domain.

Key Words: *Classical Continuum, Thin Plate Theory, Extensional Motion, Flexural Motion, Legendre Polynomial*.

INTRODUCTION
In classical continuum, modeling an object as a continuum assumes that the substance of object completely fills the space it occupies. So the model in this way ignores the fact that matter is made of atom and is not continuous.
In this model, plate theories are descriptions of the mechanics of flat plates that draws from theory of beam. Plates are defined as a structural element with a small thickness compared to the planar dimension. In plate theory we take the advantage of this disparity in length scale to reduce the three dimensional problem to two-dimensional problem. The aim of the plate theory is to calculate stress, deformation, and to study on the vibration of plate under external stimuli. There are two well-accepted plate theories which are:

Kirchoff-Love Plate Theory
This is an extension to Euler-Bernoulli beam theory and was developed by Love (1982). Main hypothesis of this theory is given in Reddy (1997) are as follows:
i. Plane normal to the mid-surface remains plane and normal to it after deformation.
ii. Thickness of the plate does not change during a deformation.

Mindlin-Reissner Theory
This model was considered by Reissner (1944), Wang *et al.,* (2001) where we should note that the plane section is remained plane but no longer perpendicular to the centroidal plane due to linear contribution of shear effect along thickness of the plate for its deformation which was ignored in previous theory. As earlier it is assumed that the thickness does not change during deformation. This implies that normal stress through the thickness is ignored. This assumption is called plane-stress condition.
Classical plate theory really develops after the pioneering work of Kirchoff. After that thousands of publications are presented which try to give the foundations and methods of deduction of Kirchoff-Love theory and its possible improvements; books of Ciarlet (1997, 2000) can be mentioned in this context. This two dimensional linear model which are in fact, the two-dimensional approximation of three dimensional theories of elastic plate involves a priori assumptions regarding the variations of unknowns (i.e. displacements and the stresses) across the thickness of the plates. The assumptions on which the theory of small deflection of thin elastic plate is based can be found in the book of Timoshenko and Woinowsky-Krieger (1985). Another method which has been used to obtain two dimensional models of thin elastic plates is the so-called asymptotic expansion method. In this method, a formal power series expansion of three-dimensional solution is used by considering the thickness of the plate as the small parameter and the Kirchoff model of linear elastic isotropic plates is obtained as the leading term of formal asymptotic expansion.

Levinson (1980, 1981) used vector-approach to formulate the equations of equilibrium of isotropic beam and plates. Reddy(1984) independently developed third-order laminate plate theory and derived equations of motion by making special assumption on displacement field using principle of virtual displacement. Bickford (1982) and Heyliger and Reddy (1988) also derived a third-order beam theory which is variationally consistent. Books of Reddy (1984, 1999) can be consulted to get an overview on this subject and for more references. Wang (1995), Wang *et al.* (2000), Wang and Kitipornchai (1999) are other significant works that are related to classical thin plate.

In this paper we have attempted to formulate a vibration problem of classical plate in some different fashion. We include terms which were ignored in most of the previous work we have seen, though incorporation of such terms do not violate the basic plate hypothesis which was made in previous works.

Classical Theory of Linear Elasticity

In case of small deformation theory of linear elasticity strain-measures are given by strain tensor ε_{ij} can be written as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{,where } i, j, k = 1,2,3 \; . \tag{1}$$

Here comma denotes the differentiation with respect to rectangular co-ordinate x_i . Also summation convention holds for the repeated indices .

In classical theory the balance of linear momentum and angular momentum are in the following form:

$$t_{ij,j} + \rho(f_i - \ddot{u}_i) = 0 \; , \; \varepsilon_{ijk} t_{jk} = 0 \tag{2}$$

where ρ is the density, f_i is the body-force density in i-direction. In the linear theory of elasticity, the equations of motion (2) are supplemented by the following constitutive equation

$$t_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij} \tag{3}$$

where λ, G correspond to Lame' constants.

Theoretical Assumptions of Plate Theory

Consider a thin plate of thickness $2h$. x_3 -axis is taken along the thickness of the plate x_1, x_2 -axes lie on the planar dimension of the plate to constitute a right-handed rectangular co-ordinate system in space having $x_3 = 0$ as its median plane. Let C denote the boundary curve of the median plane of plate S .

We introduce a new length scale by taking h as unit-length. Now with that new scaling, we make the following assumptions: the displacement vector $\vec{u}$ and the stress tensor t_{ij} can be expressed in terms of a series in Legendre polynomial $P_n(x_3)$, so that we can write

$$\vec{u} = \sum_{n=0}^{N} \vec{u}^{(n)} P_n(x_3) \text{ and } t_{kl} = \sum_{n=0}^{N} [t_{kl}^{(n)} / I_n] P_n(x_3) \tag{4}$$

where $\vec{u}^{(n)}, t_{kl}^{(n)}$ are independent of x_3.

Here we consider $-1 \le x_3 \le 1$, $P_n(x_3)$ is Legendre polynomial and $\{P_n(x_3) : n \in \mathrm{N}\}$ forms a complete orthogonal set in $-1 \le x_3 \le 1$ with respect to the inner product, which is defined by $\langle f \mid g \rangle = \int_{-1}^{1} fg \, dx$, provided f, g are L^2-measurable function in [-1,1]. We know L^2 is a norm-linear space with norm, which is denoted by ||.|| and is defined by $\|f\|^2 = < f, f >$.

We take $I_n = \|P_n\|^2 = 2/(2n+1)$.

There is a well-known result of analysis that for any function f in L^2-measurable space in [-1,1], we can have a series $\sum_{n=0}^{\infty} c_n P_n$, where $c_n =< f, P_n > / I_n$, which converges to f in L^2 sense. So if we assume that displacement field is L^2- function or continuous function, along its thickness, i.e. in, $-1 \le x_3 \le 1$ then equation (4) is admissible as per our assumption.

In plate theory, it is assumed that $t_{33} = 0$. As usual, we assume that thickness of the plate remains unchanged due to deformation, which gives a further restriction on the assumption of u_3 , will be discussed in the next section.

At first, we make no presumption on N, later we discuss the plate-problem for N=2.

Mathematical Formulation of Plate Theory

We mainly concern about the vibration of plate. We consider the plate surfaces, $x_3 = \pm 1$ are free of traction, so that flexural motion appears due to the initial stimuli and body-force density $f_i = 0$.

Now we decompose equation (2) into components along the orthogonal co-ordinate function $P_n(x_3) : n \in \mathrm{N}$ of the Hilbert-space L^2 in [-1,1], which gives

$$\left\langle t_{kl,k} - \rho \ddot{u}_l \,\middle|\, P_n(x_3) \right\rangle = 0.$$

The above equation can be written as

$$t_{Kl,K}^{(n)} + [t_{3l} P_n(x_3)]_{-1}^{+1} - \int_{-1}^{1} t_{3l} P_n^{'}(x_3) dx_3 - \rho \ddot{u}_l^{(n)} I_n = 0 ,$$

where $K \in \{1,2\};\ k, l \in \{1,2,3\}$.

Using the condition that the boundary surface $x_3 = \pm 1$ is free of traction i.e. $t_{3l}(x_1, x_2, \pm 1) = 0$ and employing properties of Legendre polynomial we finally get the following equations:

$$t_{Kl,K}^{(0)} - \rho \ddot{u}_l = 0 \tag{5.a}$$

$$\text{or, } t_{Kl,K}^{(n)} - 2 \sum_{n-2p \ge 1}^{n} \bar{t}_{3l}^{(n-2p-1)} - \rho \ddot{u}_l^{(n)} = 0 \qquad \text{if } p \ge 0,\ n \ge 1;\ p, n \in \mathrm{N} \tag{5.b}$$

,where $\bar{t}_{ij}^{(n)} = t_{ij}^{(n)} / I_n$.

where l runs over {1,2,3} and K runs over {1,2}. So through equations (5.a),(5.b) we make the 3D problem into 2D problem in which thickness co-ordinate x_3 is eliminated.

Taking into consideration $P_n(x) = (-1)^n P_n(-x)$, we note that, $u_K^{(n)}; K \in \{1,2\}$ the component of displacement field u_K along the co-ordinate function $P_n(x_3) : x_3 \in [-1,1]$ represents extensional motion or flexural motion if n is even or odd respectively and $u_3^{(n)}$ represents flexural motion $\forall n$.

From the hypothesis $t_{33} = 0$, we get $\varepsilon_{33} = \lambda / (\lambda + 2G) \varepsilon_{KK}, K \in \{1,2\}$. We use this to establish constitutive relations from (3), which are as following:

$$t_{KL}^{(n)} = \frac{EI_n}{1-\nu^2} \left[u_{,MM}^{(n)} \delta_{KL} + \frac{(1-\nu)}{2} (u_{K,L}^{(n)} + u_{L,K}^{(n)}) \right] \tag{6.a}$$

$$t_{K3}^{(n)} = t_{3K}^{(n)} = I_n G u_{3,K}^{(n)} + 2G \sum_{p \ge 0}^{n+2p+1 \le N} u_K^{(n+2p+1)} \tag{6.b}$$

K, L, M runs over {1, 2} and $p \in \mathrm{N}$, $E = G(3\lambda + 2G)/(\lambda + G)$, $\nu = \lambda / 2(\lambda + G)$.

The equations of motion are divided into two parts in which one set of equations are written for extensional motion and the other set of equations for flexural motion. It is shown with the presumptions and plate hypothesis that the two sets are uncoupled for N=1. However, for N>1, such un-coupling is not obvious and as a result one cannot distinguish the field equations in such way.

One of our plate-hypotheses is that, the thickness of the plate remains unchanged due to vibration. This gives that $\int_{-1}^{1} u_{3,3} dx_3 = [u_3]_{x_3=-1}^{1} = 0$, that means $u_3^{(n)} = 0$ if n is odd.

Substituting relation (6.a), (6.b) into relation (5.a), (5.b) we obtain the following field equation :

$$Gu_{K,LK}^{(0)} + Gu_{L,KK}^{(0)} - \rho \ddot{u}_L^{(0)} = 0 \tag{7.a}$$

$$Gu_{3,KK}^{(0)} + G\sum_{i\geq 0} u_{L,KK}^{(2p+1)} - \rho \ddot{u}_3^{(0)} = 0 \tag{7.b}$$

$$I_n Gu_{K,LK}^{(n)} + Gu_{L,KK}^{(n)} - 2\sum_{n-2i\geq 1}^{n} \bar{t}_{3L}^{(n-2p-1)} - \rho I_n \ddot{u}_L^{(n)} = 0$$

where $t_{3K}^{(n)} = Gu_{3,K}^{(n)} + 2G\sum_{i\geq 0} u_K^{(n+2p+1)}$ (7.c)

$$I_n [Gu_{3,KK}^{(n)} - \rho \ddot{u}_3^{(n)}] - 2G\sum_{i\geq 0} u_{K,K}^{(n+2p+1)} = 0 \tag{7.d}$$

where $K, L, M \in \{1,2\}$ and $p \in \mathrm{N}$.

A set of boundary and initial conditions can be derived by the decomposition along $P_n(x_3), n \in \mathrm{N}$ in [-1, 1].

A set of boundary conditions are

$t_{KL}^{(n)} n_K = t_L^{(n)}$ on C_L (8.a)

$t_{K3}^{(n)} n_K = t_3^{(n)}$ on C_L (8.b)

$u_K^{(n)} = u_{0K}^{(n)}$ on $C - C_L$ (8.c)

$u_3^{(n)} = u_{03}^{(n)}$ on $C - C_L$ (8.d)

where $C_L \subset C$.

Initial conditions of this problem are

$u_K^{(n)}(t=0) = U_{0K}^{(n)}$, on S (8.e)

$u_3^{(n)}(t=0) = U_{03}^{(n)}$ if n is even, on S (8.f)

$\dot{u}_K^{(n)}(t=0) = V_{0K}^{(n)}(x_1, x_2)$, on S (8.g)

$\dot{u}_3^{(n)}(t=0) = V_{03}^{(n)}(x_1, x_2)$ if n is even on S . (8.h)

Here the quantities on the right-hand-side is prescribed on the boundary C or on the surface S.

Case study for N=2:

Field equation for N=2 is obtained from (7.a)-(7.d), by setting $u_k^{(n)} = 0; n > 2$. Now we decompose displacement and micro-rotation in a way, which is known as Helmholtz decomposition in 3D.

Let $(u_1^{(n)}, u_2^{(n)}, 0) = \vec{\nabla}\upsilon^{(n)} + \vec{\nabla} \times \vec{V}^{(n)}$; where $\vec{V}^{(n)} = (0,0,V^{(n)})$ (9)

where $\upsilon^{(n)}, V^{(n)}$ are all function of x_1, x_2, t ; $n \in \{0,1,2\}$.

Now substituting (9) into the field equations for N=2 and separating the curl and gradient part of the equations we get the equations as following:

$$[EI_1/(1-\nu^2)]\nabla^2\upsilon^{(1)} - 2G\upsilon^{(1)} - 2Gu_3^{(0)} + \rho I_1 \ddot{\upsilon}^{(1)} = 0 \tag{10.a}$$

$$GI_1\nabla^2 V^{(1)} - 2GV^{(1)} - \rho I_1 \ddot{V}^{(1)} = 0 \tag{10.b}$$

$$[EI_2/(1-\nu^2)]\nabla^2\upsilon^{(2)} - \frac{4}{I_1}G\upsilon^{(2)} - \rho I_2\ddot{\upsilon}^{(2)} = 0 \tag{10.c}$$

$$GI_2\nabla^2 V^{(2)} - \frac{4}{I_1}GV^{(2)} - \rho I_2\ddot{V}^{(2)} = 0 \tag{10.d}$$

$$[E/(1-\nu^2)]\nabla^2\upsilon^{(0)} - \rho\ddot{\upsilon}^{(0)} = 0 \tag{10.e}$$

$$G\nabla^{(2)}V^{(0)} - \rho\ddot{V}^{(0)} = 0 \tag{10.f}$$

$$G\nabla^2 u_3^{(0)} + G\nabla^2\upsilon^{(1)} - \rho\ddot{u}_3^{(0)} = 0 \tag{10.g}$$

$$G\nabla^2 u_3^{(2)} - \rho\ddot{u}_3^{(2)} = 0 \tag{10.h}$$

With the help of (9) we get the following relation :

$$\bar{t}_{KL}^{(n)} = [E\nu/(1-\nu^2)]\nabla^2\upsilon^{(n)} + (1-\nu)\upsilon_{,KL}^{(n)} + G\varepsilon_{KM3}V_{,ML}^{(n)} + G\varepsilon_{LM3}V_{,MK}^{(n)}$$

$$\bar{t}_{K3}^{(n)} = Gu_{3,K}^{(n)} + \frac{2}{I_n}G\upsilon_{,L}^{(n+1)} + \frac{2}{I_n}GV_{,M}^{(n)}\varepsilon_{KM3}$$

where $K, L, M \in \{1,2\}$ and $n = 0,1,2$. With the help of above relations, the boundary and initial conditions (8.a)-(8.h) can be expressed in terms of $\upsilon^{(n)}, V^{(n)}$ and thus they can be made in suitable form to represent the boundary conditions and initial conditions for (10.a)-(10.h).

Passage to the classical plate-theory

From equations (10.a) and (10.g) if we eliminate $\upsilon^{(1)}$, we get

$$-[EI_1/(1-\nu^2)]\nabla^4 u_3^{(0)} + \rho\left(\frac{EI_1/(1-\nu^2)}{G} + I_1\right)\nabla^2\ddot{u}_3^{(0)} - \frac{\rho^2 I_1}{G}\frac{\partial^4}{\partial t^4}u_3^{(0)} - 2\rho\ddot{u}_3^{(0)} = 0 \tag{11}$$

Now equations are derived by introduction of new length-scale by taking h as unit length, which is mentioned earlier .

Now we want to consider the effect of thickness h of the plate in the dynamics of plate. So we now go back to original length-scale so that we have

$x_i = \bar{x}_i / h$, $\bar{x}_i$ is the co-ordinate in original scale, where $i = 1,2,3.$

$(\partial/\partial x_i) \equiv h(\partial/\partial\bar{x}_i).$

Let, $\mathrm{B} = EI_1/(1-\nu^2)$ and $D = D_h = D(h) = \mathrm{B}h^3.$

In dimension of $EI_1/(1-\nu^2)$, length-dimension appears through $[Length]^{-1}$.

Thus we have $D_1 = h\overline{D}_1$, where $\overline{D}_1$ is the quantity $EI_1/(1-\nu^2)$, measured in original length-scale.

Similarly, we have $\rho = h^3\bar{\rho}$, $\nabla^n = h^n\overline{\nabla}^n ; n \in \mathrm{N}$.

Now if we set $G \to \infty$ i.e. we ignore shear deformation in (11), we get

$$-D_1\nabla^4 u_3^{(0)} + 2\rho I_1\nabla^2\ddot{u}_3^{(0)} - 2\rho\ddot{u}_3^{(0)} = 0$$

If we re-write the above equation in original length-scale, it takes the form

$$-\overline{D}_1 h.h^4\overline{\nabla}^4 u_3^{(0)} + +2I_1\bar{\rho}h^3.h^2\overline{\nabla}^2\ddot{u}_3^{(0)} - 2\bar{\rho}h^3\ddot{u}_3^{(0)} = 0$$

Now cancelling h^2 and omitting bar-notation in (11), we get the form of equation in original length-scale as following

$$-D_1 h^3\nabla^4 u_3^{(0)} + 2\rho I_1 h^3\nabla^2\ddot{u}_3^{(0)} - 2\rho h\ddot{u}_3^{(0)} = 0 \tag{12.a}$$

Other plate-equations of (10.a)-(10.h) in original length-scale takes the form

$$\frac{D_1 h^3}{G}\nabla^2\upsilon^{(1)} - 2h\upsilon^{(1)} - 2hu_3^{(0)} + \frac{h^3\rho I_1}{G}\ddot{\upsilon}^{(1)} = 0 \quad (12.b)$$

$$I_1 h^3\nabla^2 V^{(1)} - 2hV^{(1)} - \frac{\rho I_1 h^3}{G}\ddot{V}^{(1)} = 0 \quad (12.c)$$

$$\frac{h^3 E I_2}{(1-\nu^2)}.\frac{\nabla^2\upsilon^{(2)}}{G} - \frac{4}{I_1}h\upsilon^{(2)} - \frac{h^3\rho I_2}{G}\ddot{\upsilon}^{(2)} = 0 \quad (12.d)$$

$$h^3 I_2\nabla^2 V^{(2)} - \frac{4}{I_1}hV^{(2)} - \rho h^3 I_2\ddot{V}^{(2)} = 0 \quad (12.e)$$

$$\frac{E}{(1-\nu^2)}\nabla^2\upsilon^{(0)} - \rho\ddot{\upsilon}^{(0)} = 0 \quad (12.f)$$

$$\nabla^2 V^{(0)} - \frac{\rho}{G}\ddot{V}^{(0)} = 0 \quad (12.g)$$

$$\nabla^2 u_3^{(2)} - \frac{\rho}{G}\ddot{u}_3^{(2)} = 0 \quad (12.h)$$

Now if we neglect shear-deformation and rotatory inertia, i.e. $G \to \infty,\ h^3 \to 0$, such that Gh^3, D are finite and set of equations (12.a)-(12.h) reduce to the following equations :

$$-D\nabla^4 u_3^{(0)} - 2\rho h\ddot{u}_3^{(0)} = 0 \quad (13.a)$$

$$\upsilon^{(1)} + u_3^{(0)} = 0 \quad (13.b)$$

$$\nabla^2 u_3^{(2)} = \nabla^2\upsilon^{(0)} = \nabla^2 V^{(0)} = V^{(1)} = \upsilon^{(2)} = V^{(2)} = 0 \quad (13.c)$$

which agrees with the plate equation according to classical theory.

If we consider a load $P/2h$ along the direction of x_3-axis and replace $\rho\ddot{u}_3^{(0)}$ in (13.a) by $\rho\ddot{u}_3^{(0)} - P/2h$ then we obtain [By the application of D'Alambert's method (incorporation of pseudo-force in constructing equation of dynamics)] the following plate equation:

$-D\nabla^4 u_3^{(0)} - 2\rho h\ddot{u}_3^{(0)} + P = 0$ which is well-known plate equation in classical theory [Kirchoff-Love theory] in the case of negligible rotatory inertia and shear-deformation.

From (13.c) we get $V^{(1)} = 0$, which implies $\vec{u}^{(1)} = \vec{\nabla}\upsilon^{(1)} = -\vec{\nabla}u_3^{(0)}$. Using this result, we obtain from (4) for N=1,

$$u_1 = u_1^{(0)} - x_3 u_{3,1}^{(0)}$$

$$u_2 = u_2^{(0)} - x_3 u_{3,2}^{(0)}$$

Expression of u_1, u_2 agrees with the consideration of the expression for u_1, u_2 in Kirchoff-Love Plate theory.

Conclusion

So in this article we have looked into the plate problem in a new fashion by incorporating more terms that were ignored in earlier theories. We re-defined the plate problem in a new approach and it has been shown that the equations derived in this paper agrees with that of Kirchoff-Love plate theory. So that our formulation can be taken as an extension to Kirchoff-Love plate theory and it is done in some different fashion.

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