International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2013 Vol. 3 (2) April-June, pp.87-98/Sakwa et al.

Research Article

THERMODYNAMICS OF A GRAND-CANONICAL BINARY SYSTEM AT LOW TEMPERATURES

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ABSTRACT
The ground state properties of trapped atomic Boson-Fermion mixture at near absolute zero temperature Kelvin is studied using second quantization techniques. An effective Hamiltonian for the Binary system is developed in terms of the magnitude of fluctuations to bring out the interplay between boson-boson, boson-fermion & fermion-fermion interactions and their implications on the thermodynamics properties of the system. The study focused on a Grand Canonical Binary system of ^{3}He-^{4}He isotopes whose thermodynamic properties have been determined by distinctively singling out the boson-boson, boson-fermion, and fermion-fermion interactions from which the energy density is established algebraically. The specific heat and the entropy of the system were consequently established and analyzed. The total energy of the system is found to increase with increase in occupation number of the system. The jump in the specific heat seems to suggest a phase-like transition at temperature of about 0.4K. Entropy decreases with temperatures.

Key Words: *Bose-Einstein Condensation, Binary Condensates, Grand Canonical Ensemble.*

INTRODUCTION

Since the first realizations of Bose-Einstein Condensation (BEC) in trapped dilute atomic gases (Inguscio *et al.*, 1998), this lively field of research has generated impressive experimental results illuminating basic quantum phenomena. Besides the studies using bosonic atoms first, results were obtained on the cooling of fermionic atoms to a low temperature regime where quantum effects dominate the properties of the gas (Demarco and Jin 1999). One of the exciting prospects is the observation of Bardeen, Cooper and Schrieffer (BCS) transition of the degenerate Fermi gases to superfluid state. Binary condensates that were first realized in 1995 for Rubidium (Anderson *et al.,* 1995) Sodium (Bradley *et al.,* 1995) and Lithium (Davis *et al.,* 1995) provide unique opportunities for exploring quantum phenomena on a macroscopic scale. These systems differ from ordinary gases, liquids and solids in terms of the particle density which is low and the temperature must be of order 10^{-5} Kelvin or less to observe quantum phenomena.

The BEC in the alkali gases were only possible due to trapping and cooling techniques, which can be created in an almost pure form. BEC's have been realized with all alkali gases, except francium, and with hydrogen and chromium (Griesmaier *et al.,* 2005) as well.

From other works on the trapped binary bose-fermion system, the theoretical description of the system has been developed in the mean-field approximation to determine the boson and fermion density profiles at zero temperature (Mølmer *et al.,* 1998, Amoruso *et al.,* 1998, Miyakawa *et al.,* 2000) and the related properties of stability against phase separation and collapse, numerically(Nygaard and Molmer, 1999), (Roth and Feldmeier, 2002) and (Akdeniz *et al.,* 2002) and by a Gaussian Variation Ansatz (Yi and Sun 2001). Other works performed on the strongly interacting mixture of ^{4}He - ^{3}He was on the calculation of the structural factors (Al-Hayek and Tanatar, 1999; Mazzanti *et al.,* 2000). The works on alkali earth metal atoms was done with help of mean field approximation while variation approaches were used for strongly –interacting helium. Other authors used methods such as exact solution using Bethe Ansatz (Imambekov and Demler, 2006) bosonization techniques (Mathey *et al.*, 2007) quantum Monte Carlo (Pollet *et al.*,2006) and T-matrix approximation (Barillier *et al.*, 2007, Sakwa and Khanna, 2000).

Recently new mixtures have been realized in the lab including ^{6}Li-$^{87/85}$Rb (Deh *et al.*, 2010) and ^{40}K-^{41}K (Zeng-Qiang Yu *et al.*, 2011) using mean field treatment of a two channel model (Powell *et al.*, 2008). Some recent works studied boson-fermion pairing effects within a single channel model for broad resonance (Song *et al.*, 2010).

Statistical thermodynamics explains the thermodynamic behavior of macroscopic systems as derived from microscopic properties of the constituents. (Ayodo 2008, Ayodo *et al.*, 2010), used the statistical approach to study the thermodynamic properties of ^{7}Li-^{6}Li and potassium mixtures. The effects of particle interactions on the stability of the bose-fermi mixtures were studied and found that for Lithium bose-fermi system, there is a discontinuity at the centre of the trapping potential and for pottassim bose-fermi system there was a critical condensate radius of 6 oscillator units with the system moving spontaneously from negative attractive regime to positive attractive regimes.

Quantum thermodynamic perturbative theory for many body system in which the particles interact via a pair potential containing short ranged and very large repulsive part was developed by (Khanna *et al.*, 2011). Particles at low densities were studied and their effects on the ^{4}He. The particle densities decrease with increase in hardsphere diameters for a fixed saturation density.

A lot of studies that have been done on boson-fermion mixtures using many different approaches to find particle interactions (Molmer *et al.*, 1998, Amoruso *et al.*, 1998, Miyawaka *et al.*, 2000), stability, collapse against mean field numerically (Nygaard and Molmer, 1999), (Roth and Feldmeier 2002) and (Akdeniz *et al.*, 2002 and Yi and Sun 2001), structural factors (Al-Hayek and Tanatar, 1999; Mazzanti *et al.*, 2000)and the thermodynamic properties of these systems by (Ayodo *et al.*, 2010). The main focus has been statistical mechanics and related approaches that were used in determining these properties of the boson-fermion system. Second quantization techniques so far have not been embraced as a method in the study of binary boson-fermion systems in the vicinity of zero Kelvin temperature. Particle interactions between boson-boson, boson-fermion and fermion-fermion and have not been studied using 2nd quantization techniques.

This study enhances the understanding of the ground state properties of binary mixtures in the vicinity of absolute zero temperatures using second quantization techniques taking into consideration the many body interactions. We have investigated the ground state properties of trapped atomic Boson-Fermion mixture using the effective Hamiltonian for Bose-fermion (Roth and Feldmeier, 2002) and further investigate the interplay between boson-boson, boson-fermion and fermion-fermion interactions and their implications on the energy and transition properties of the grand canonical binary mixture using second quantization formalism. The total energy of the grand canonical binary system of ^{3}He-^{4}He was determined and thereafter the transition properties of the system determined respectively.

2. Theoretical Formulation

2.1. Introduction

To describe the properties of the binary boson-fermion mixture at zero temperature, the energy functional in mean-field approximation is constructed. The atoms can be considered as inert interacting particles, i.e. internal excitations of the atoms are not relevant.The many-body state ψ describing the system is the product of a bosonic N_b-body state $|\psi_b>$and a fermionic N_f-body state$|\psi_f>$. Thus

$$\psi = |\psi_b > | \psi_f > \tag{1}$$

Where

$$|\psi_b >= |\psi_1 > |\psi_2 > \cdots |\psi_{nb} > = \prod_{i=1}^{N_b} |\psi_{bi} \tag{2}$$

Equation (2) is the product of N_b identical single particle states $|\psi_{bi}>$ which is symmetric for bosons. The fermionic state is a Slater determinant i.e. an anti-symmetrized product of different single particle state given as

$$|\psi_f>= A\prod_{i=1}^{N_f}|\psi_{fi}> \quad (3)$$

where A is the anti-symmetrization operator.

In this scenario pair interaction is weak hence the mean field approximation is a good projection. The two body potential has an attractive part at a larger particle distance and strong repulsion at short distances.

A full atom-atom potential is replaced by a suitable effective interaction potential with relevant physical properties of the original potential (Roth and Feldmeier, 2001). Due to the large inter-atomic distance the atom-atom interaction can in general be described by an effective contact interaction for all partial waves (Roth and Feldmeier 2001,2002), using the effective contact interaction one can set up the Hamiltonian of binary boson-fermion mixture. It consists of the bosonic component (B), the fermionic component (F) and a part which represents the interaction between the two species (BF). The Hamiltonian of the interacting mixture reads:

$$H = H_{bb} + H_{ff} + H_{bf} \quad (4)$$

The s-wave interaction between two bosons is described by the first term of Eq. (4); since we consider a pure Bose-Einstein condensate at zero temperature only the s-wave term is needed. Higher even partial waves are negligible. For the interaction between a bosonic and a fermionic atom s- and p-wave terms contribute. The operator of the s-wave boson-fermion contact interaction forms the last term of the Hamiltonian (2). Since the s-wave interaction dominates in many cases of interest, we will neglect the p-wave interaction for this discussion. For identical fermions s-wave contact interactions are prohibited by the Pauli principle. However, p-wave interactions can have significant influence on the structure and stability of the fermionic components (Roth and Feldmeier 2001, 2002).

The effective Hamiltonian as drawn from equation (4) can now be used to study the ground state properties of boson-fermion mixture at near zero temperature. The expectation value of the Hamiltonian eq. (4) calculated with the many-body state defines the total energy of the mixture as

$$E_T =<\psi|H|\psi>=<\psi_b|H_{bb}|\psi_b>+<\psi_f|H_{ff}|\psi_f>+<\psi_{bf}|H_{bf}|\psi_{bf}>$$

$$E_T = E_{bb} + E_{ff} + E_{bf} \quad (5)$$

where ψ is defined in equation (1). Equation (5) is the total energy of the binary mixture.

The total energy E_T is decomposed into a purely bosonic part (B), a fermionic part (F) and the interaction part between the two species (BF). In order to keep the discussion simple we restrict ourselves to symmetric systems with equal numbers of bosons and fermions

$N_b = N_f$.

2.2. Dilute Boson-Fermion Mixtures

In the theory of a trapped Bose-Fermi mixture, the dilute mixture is treated as thermodynamic equilibrium system under the grant canonical ensemble whose thermodynamic variables N_B and N_F are respectively the total number of trapped bosonic and fermionic atoms, T is the absolute temperature, and μ_B and μ_F the chemical potentials for boson and fermion respectively. The density Hamiltonian of the system is given by equation (4). The distribution numbers must satisfy the following relations

$$\mathrm{N} = \mathrm{N}_f + \mathrm{N}_b = \sum_{j=1}^{\infty} n_{fj} + \sum_{j=1}^{\infty} n_{bj} \quad (6)$$

$$E = E_b + E_f = \sum_{j=1}^{\infty} n_{bj}\,\epsilon_j + \sum_{j=1}^{\infty} n_{fj}\,\epsilon_j$$

where E_f and E_b are the total internal energies of fermions and bosons in the mixture.

2.2.1. Boson Interaction

The energy levels of anharmonic oscillator in one dimension are determined. The harmonic oscillator of unperturbed Hamiltonian in one dimension is written as,

$$H = \frac{P^2}{2m} + \frac{1}{2}kx^2 \tag{7}$$

where $k = m\omega^2$, P is the momentum, m is the mass of bosons, x is the displacement vector and ω is the angular velocity of the oscillator.

The following perturbation equation is added to the harmonic oscillator (Khanna *et al*., 2010)

$$V_{bb} = \beta x^3 + \gamma x^4 \tag{8}$$

Where β and γ are constants of perturbation, x is displacement vector.

The perturbed Hamiltonian for bosons H_{bb} becomes,

$$H_{bb} = H_{obb} + V_{bb} \tag{9}$$

where $H_{obb} =$ unperturbed Hamiltonian and V_{bb} = perturbation potential that causes anharmonicity in the harmonic interaction as given by equation (8). Hence

$$H_{bb} = \frac{P^2}{2m_{bb}} + \frac{1}{2}kx_{bb}^2 + (\beta x^3 + \gamma x^4) \tag{10}$$

The displacement vector x can be defined in terms of creation a^+ and annihilation a operators, obeying the commutation relations, such that

$$x = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a + a^+) \tag{11}$$

$$P = i\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}(a - a^+) \tag{12}$$

Using equation (11) and (2) in equation (7) gives

$$H_{obb} = \left[\frac{\hbar\omega}{2}(2n+1)\right]$$

Hence, the unperturbed Hamiltonian for the system may be written as

$$H_{obb} = \sum_k \hbar\omega\left(a_k^+ a_k + \frac{1}{2}\right) \tag{13}$$

For N identical non-relativistic particles

$$|\psi> = |\psi(x_1 x_2 \dots \dots x_N)> \tag{14}$$

and the perturbation potential given by equation (8) can be written in terms of operators as

$$V_{bb} = \beta\left(\frac{\hbar}{2m_{bb}\omega}\right)^{\frac{3}{2}}(a + a^+)^3 + \gamma\left(\frac{\hbar}{2m_{bb}\omega}\right)^2(a + a^+)^4 \tag{15}$$

The resultant H_{bb} is

$$H_{bb} = \sum_{i=1}^{A}\hbar\omega\left(a_i^+ a + \frac{1}{2}\right) + \sum_{i=1}^{n}\left[\beta\left(\frac{\hbar}{2m_{bb}\omega_i}\right)^{\frac{3}{2}}(a_{ib} + a_{ib}^+)^3 + \gamma\left(\frac{\hbar}{2m_{bb}\omega_i}\right)^2(a_{ib} + a_{ib}^+)^4\right] \tag{16}$$

The equation can be expanded in terms of aa^+ and then replaced by the equivalent of n such that:

$$H_{bb} = \sum_{i=1}^{A}\hbar\omega\left(a_i^+ a + \frac{1}{2}\right) + \sum_{i=1}^{n}\left[\beta^2\left(\frac{\hbar}{2m_{bb}\omega_i}\right)^3 18n^3 + \gamma\left(\frac{\hbar}{2m_{bb}\omega_i}\right)^2(6n^2 + 6n + 3)\right] \tag{17}$$

The first summation is the zero energy of the system; second summation is the perturbation potential due to interaction of boson pairing. It's this term that causes the anharmonicity in a system of bosons.

2.2.2. Fermion Interaction

The general expression for the Hamiltonian of a system of free particles is given as

$$H = \sum_i \mathcal{E}_i n_i = \sum_i \mathcal{E}_i a_i^+ a_i \quad (18)$$

The number of particles in n_i is constant and the total number of particles N_o in the system is conserved. Note that Fermions obey the anti- commutation laws. For an interacting system of fermions; the Hamiltonian H_{ff} conserves the total number of particles. The fockspace picture of the many- body problem is equivalent to the grand canonical ensemble of statistical mechanics. Thus instead of fixing the number of particles, langrage multiplier λ is introduced to weigh contributions from different parts of the fockspace. Thus the Hamiltonian operator is defined as

$$H_{ff} = \sum \varepsilon_i n_i - \sum \lambda n_i = \sum(\varepsilon_i - \lambda) n_i \quad (19)$$

In Hilbert space with fixed number of particles this amounts to a shift of the energy by λn. The system is now allowed to choose the sector of the fockspace but with requirement that the average number of particles $< n >$ is fixed to some number

n_o. In this thermodynamic limit as $(n \to \infty)$; λ the chemical potential of fermions represents the difference of the ground state energies between two sectors with n+1 and n particles .The value of λ will be fixed by the requirement,

$$< n > = n_o \quad (20)$$

The choice of λ determines only the average value of n_o in the λ-system. The fluctuation of n_o in this system however is very small for large n.

Using equation (18) in equation (19), the perturbed Hamiltonian for a system of fermions in terms of operators is obtained as

$$H_{ff} = \sum_{i=1}^{A} \hbar\omega a_i^+ a_i - \lambda(a_i^+ a_i) \quad (21)$$

2.2.3 Bosons-Fermions

A hybrid system of boson-fermion is anharmonic. The anharmonic system in one dimension may be expressed as (Samiha 2000)

$$H_{bf} = H_o + \lambda x^4 \quad (22)$$

where λ is the perturbation parameter.

The unperturbed Hamiltonian for a system of boson-fermion is obtained as

$$H_{obf} = \sum_i \hbar\omega \left(a_i^+ a_i + \frac{1}{2}\right) \quad (23)$$

Substituting equation (11) into equation (22), the perturbation potential becomes

$$\lambda x^4 = \lambda \left[\left(\frac{\hbar}{2\mu_{bf}\,\omega}\right)^{\frac{1}{2}} (a_i + a_i^+) \right]^4 \quad (24)$$

Since it is a mixed system of boson-fermion the reduced mass is obtained as

$$\mu_{bf} = \frac{m_b m_f}{m_b + m_f} \quad (25)$$

adding equation (23) and equation (24), the perturbed Hamiltonian for the mixed system of boson-fermion becomes

$$H_{bf} = \sum_i \left[\hbar\omega \left(a_i^+ a_i + \frac{1}{2}\right) + \lambda \left\{ \left(\frac{\hbar}{2\mu_{bf}\,\omega}\right)^{\frac{1}{2}} (a_i + a_i^+) \right\}^4 \right] \quad (26)$$

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Similarly the curly bracket $(a_{ib}+a_{ib}^{+})^4$ can be expanded, simplified and replaced by its equivalent of n and the resultant perturbed Hamiltonian for the mixed species H_{bf} obtained as

$$H_{bf} = \sum_i \hbar\omega\left(a_i^{+}a_i+\frac{1}{2}\right)+\lambda\left[\left(\frac{\hbar}{2\mu_{bf}\,\omega}\right)^2 (6n^2+6n+3)\right] \quad (27)$$

The first term is the zero energy of the system; second term is the interaction potential for bosons and fermions. It's this term that causes the anharmonicity of the otherwise harmonic interaction.

The total effective Hamiltonian for the binary atomic mixture from equation (4) becomes

$$H_{eff} = \sum_{i=1}^{A}\hbar\omega\left(n+\frac{1}{2}\right)+\sum_{i=1}^{n}\left[\beta^2\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^3 18n^3+\gamma\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^2 (6n^2+6n+3)\right] + \sum_i \hbar\omega\left(n+\frac{1}{2}\right)+\lambda\left[\left(\frac{\hbar}{2\mu_{bf}\,\omega}\right)^2 (6n^2+6n+3)\right]+\sum_i(\varepsilon_i-\lambda)n \quad (28)$$

2.3. Energy Theorem of the Binary System

Perturbation theory is applied to find the energy levels of anharmonic oscillator.

The eigenvalues and eigenfunctions of the unperturbed harmonic oscillator Hamiltonian, H_O is well known. The unperturbed state |ϕ> can be written as |n> since it is characterized by its energy $\left(n+\frac{1}{2}\right)\hbar\omega$

2.3.1. Bosons

The perturbed Hamiltonian for a system of bosons is given by equation (9).

The energy eigenvalue of the unperturbed Hamiltonian for bosons is given as

$$H_{ob}|n>=E_{nb}^{0}|n> = \left(n+\frac{1}{2}\right)\hbar\omega|n> \quad (29)$$

When the system is perturbed $H_b'|n>=E_b'|n>$ is the Eigenvalue problem that needs to be solved. The perturbed energy for a system of bosons to second order in terms of V_{bb} becomes

$$E_{bb} = \left(n+\frac{1}{2}\right)\hbar\omega+<n|V_{bb}|n>+<n\left|V_{bb}\frac{1}{E_n-H_o}V_{bb}\right|n> \quad (30)$$

First term in equation (30) is the zero energy, second term is the first order energy change and the third term is the second order energy. The value of V_{bb} is substituted in equation (30) we obtain.

$$E_{bb} = \left(n+\frac{1}{2}\right)\hbar\omega+<n|\sum_{i=1}^{n}\left[\gamma\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^2 (6n^2+6n+3)\right]|n>+<n|\sum_{i=1}^{n}\beta^2\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^3 18n^3\frac{1}{E_n-H_o}|n>$$

We determined the perturbed energy due to bosons as (31)

$$E_{bb} = \left(n+\frac{1}{2}\right)\hbar\omega+\gamma\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^2 (6n^2+6n+3)-\frac{18\hbar^2\beta^2}{8m^3\omega^4}n(n+1)^2$$

where n=0,1, 2, 3……. (32)

2.3.2 Fermions

The energy eigenvalue of the unperturbed Hamiltonian for fermions is given as

$$H_{of}|n>=E_{nf}^{0}|n> = \left(n+\frac{1}{2}\right)\hbar\omega|n> \quad (33)$$

When the system is perturbed the eigenvalue problem to be solved for fermions becomes

$$H_{ff}'|n>=E_{ff}'|n>=-\lambda\left(a_{ff}^{+}a\right)|n> \quad (34)$$

Adding equation (33) and equation (34) gives us the perturbed energy of the fermion system as

$$H_{ff}|n>=(n+\tfrac{1}{2})\hbar\omega|n>-\lambda\left(a_f^{+}a\right)|n> \quad (35)$$

$$E_{ff} = \left(n+\frac{1}{2}\right)\hbar\omega - \lambda n \qquad (36)$$

2.3.3 Mixed Boson-Fermion

The energy eigenvalue of the unperturbed Hamiltonian for boson-fermion mixed system is given by

$$H_{obf}\,|n>= E^{o}_{nbf}\,|n>= \left(n+\frac{1}{2}\right)\hbar\omega|n> \qquad (37)$$

when the system is perturbed , the Eigenvalue problem to be solved is

$$H'_{fb}\,|n>= E'_{fb}\,|n> \qquad (38)$$

the perturbation potential for the mixed system can be written as

$$E'_{fb} =< n\left|\lambda\frac{\hbar^2}{4\mu^2\omega^2}(a+a^{+})^4\right|n> \qquad (39)$$

Similarly expanding the term $(a+a^{+})^4$ and replacing with its equivalent in terms of n gives the result as

$$E'_{fb} = \lambda\left(\frac{\hbar^2}{4\mu^2\omega^2}\right)(6n^2+6n+3) \qquad (40)$$

from the equation

$$E_{bf} = E^{o}_{nbf} + E'_{fb} \qquad (41)$$

$$E_{bf} = \left(n+\frac{1}{2}\right)\hbar\omega + \lambda\left(\frac{\hbar^2}{4\mu^2\omega^2}\right)(6n^2+6n+3) \qquad (42)$$

The total energy of the grand canonical binary system can be obtained by adding equations (31), equation (36) and equation (42)

$$E_T = \left(n+\frac{1}{2}\right)3\hbar\omega + \gamma\left(\frac{\hbar}{2m_{bb}\,\omega_i}\right)^2(6n^2+6n+3) - \frac{18\hbar^2\beta^2}{8m^3\omega^4}n(n+1)^2 + \lambda\left(\frac{\hbar^2}{4\mu_{bf}^2\,\omega^2}\right)(6n^2+6n+3) - \lambda(n) \qquad (43)$$

2.4 Specific Heat and Transition Temperature

At the transition temperature, the probability that a normal mode of angular frequency has n_k phonons at temperature T can be written as

$$\rho_n = exp^{\left(\frac{-\Delta E_T}{\kappa T}\right)} \qquad (44)$$

where E_T, is the total energy of binary system given by equation (43) and
the normalization constant is obtained from

$$\sum_n \rho_n = 1 \qquad (45)$$

Hence the expectation value of total energy can be expressed as follows

e

$$E_n = \left(n+\frac{1}{2}\right)3\hbar\omega + \left\{\begin{matrix}\frac{3}{2}\gamma\left(\frac{\hbar}{m_{bb}\,\omega_i}\right)^2\left(n^2+n+\frac{1}{2}\right) - \frac{18\hbar^2\beta^2}{8m_{bb}^3\,\omega^4}(n^3+2n^2+n) \\ + \\ \frac{3}{2}\lambda\left(\frac{\hbar^2}{\mu_{bf}^2\,\omega^2}\right)\left(n^2+n+\frac{1}{2}\right) - \lambda n\end{matrix}\right\} e^{\left(\frac{-\Delta E_T}{\kappa T}\right)} \qquad (46)$$

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The specific heat capacity can be determined using the following relation

$$C = \frac{\partial E_n}{\partial T} \tag{47}$$

hence

$$C = \left\{ \begin{matrix} \frac{3}{2}\gamma\left(\frac{\hbar}{m_{bb}\,\omega_i}\right)^2\left(n^2+n+\frac{1}{2}\right) - \frac{18\hbar^2\beta^2}{8m_{bb}^3\,\omega^4}(n^3+2n^2+n) \\ + \\ \frac{3}{2}\lambda\left(\frac{\hbar^2}{\mu_{bf}^2\,\omega^2}\right)\left(n^2+n+\frac{1}{2}\right) - \lambda(n) \end{matrix} \right\} \frac{\hbar\omega}{kT^2} e^{\left(\frac{-\hbar\omega}{\kappa T}\right)} \tag{48}$$

The transition temperature T_C of the system is obtained from the condition that:

$$\left(\frac{\partial C}{\partial T}\right)_{T=T_c} = 0 \tag{49}$$

$$\frac{-2}{T_c^3} + \frac{\hbar\omega}{K T_c^4} = 0 \tag{50}$$

Making T_c the subject

$$T_c = \frac{\hbar\omega}{2k} \tag{51}$$

where T_C –critical temperature, $\hbar =$ Planck's constant, $\omega =$ Harmonic oscillator frequency $k=$ Boltzmann constant.

2.5. Entropy(S)

The expression relating entropy S to temperature T is given by $\partial S = \frac{\partial Q}{T}$ or

$$S_2 - S_1 = \int_1^2 dS = \int_1^2 \frac{dQ}{T} = \int_1^2 \frac{mCdT}{T} \tag{52}$$

The entropy of the system then determined to be

$$S(T) = \left\{ \begin{matrix} \frac{3}{2}\gamma\left(\frac{\hbar}{m_{bb}\,\omega_i}\right)^2\left(n^2+n+\frac{1}{2}\right) - \frac{18\hbar^2\beta^2}{8m_{bb}^3\,\omega^4}(n^3+2n^2+n) \\ + \\ \frac{3}{2}\lambda\left(\frac{\hbar^2}{\mu_{bf}^2\,\omega^2}\right)\left(n^2+n+\frac{1}{2}\right) - \lambda(n) \end{matrix} \right\} \frac{1}{T} e\left(\frac{\hbar\omega}{\kappa T}\right) + \frac{\kappa}{\hbar\omega} e^{-\left(\frac{\Delta\hbar\omega}{kT}\right)} \tag{53}$$

2.6. Essential Parameters for Data Analysis

Since $\beta(x^3)$ and $\gamma(x^4)$ must have dimensions of energy ML^2T^{-2}. The dimensions of β and γ should be $ML^{-1}T^{-2}$ and $ML^{-2}T^{-2}$ respectively, since x which is the displacement has the dimension of length L. Therefore a parameter a_o which is assumed to be fundamental to the perturbation parameters γ and β has been introduced. This parameter a_o is defined as the scattering length between boson-boson. The scattering length is taken as: $a_o = 1.3 \times 10^{-13} A^{\frac{1}{3}}\, cm$, (Khanna et al,2010) where A is the mass number. The perturbation parameters can therefore be defined as:

$\beta = \frac{\hbar\omega}{a_o^3}$, $\gamma = \frac{\hbar\omega}{a_o^4}$, $\lambda = \frac{\pi^2\hbar^2}{2m}\left(\frac{3N_A}{\pi V}\right)^{\frac{2}{3}}$,where m is the molar mass, V is the molar volume and N_A is the number of particles in one mole, that is Avogadro's number whose value is 6.02544×10^{23} mol^{-1} . The following values for different physical quantities have been used.

$\frac{plancks\ \ constant}{2\pi} = \hbar$, is given as 1.054x10^{-27} erg-s, Boltzman's constant k_B is given as 8.167x10^{-5} eV, and $\frac{the\ angular\ \ frequency}{2\pi}(\omega) = 6 \times 10^{22} s^{-1}$.

The experimental works by Chan *et al.*, (1992, 1996)and Wilks and Bett ,(1994) mainly focused on molar quantities of helium-3 and helium-4,the molar mass of helium-3 and helium -4 are 2.80g and 3.92g respectively. Boson-fermion reduced molar mass 1.63x10^{-3} kg, n is the occupation number given as 0, 1,2…,Boson molar mass m_b as 3.92x10 $^{-3}$ kg. Fermion molar mass m_f as 2.80x10 $^{-3}$kg, Boson chemical potential as 6.215x10 $^{-28}$ eV, and Fermion chemical potential as 3.184x10^{-27}eV.

RESULTS AND DISCUSSION

Results

Equation (46) was used to compute the values of internal energy in response to changes in the occupation number of states. Total energy vs occupation number was studied and found to be linear as shown in figure 1.the zero point energy is found to be 0.5j.

Figure 1: Variation of Total Energy vs against the occupation number of states

Using equation (48),the variation of specific heat versus temperature were studied and found to have a peak or turning point in the vicinity of T=0.5k

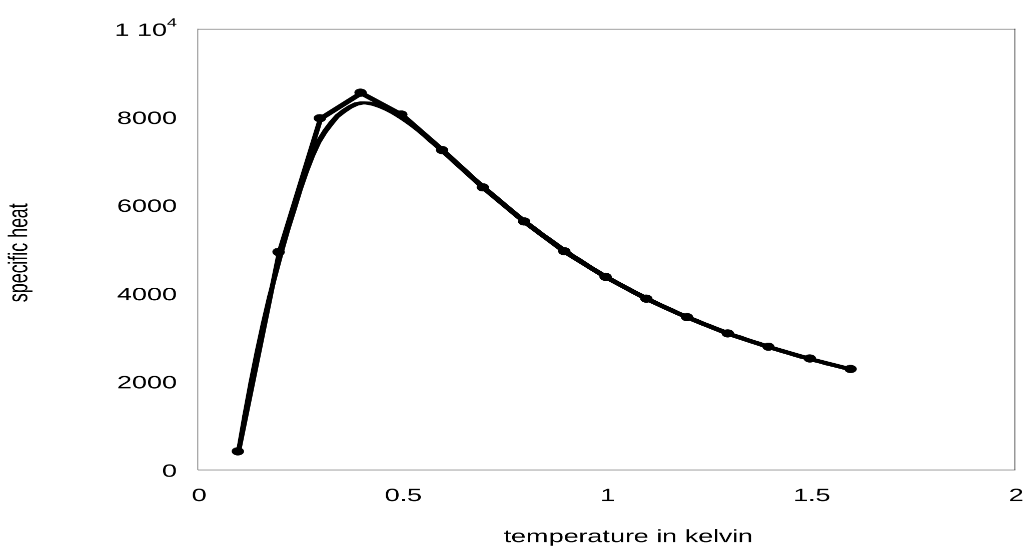

Figure 2: The graph of specific heat against temperature

Equation (53) the variation of Entropy with temperature of the binary mixture, is found to be a curve with a gently decreasing slope nearly saturating at 1.3k.

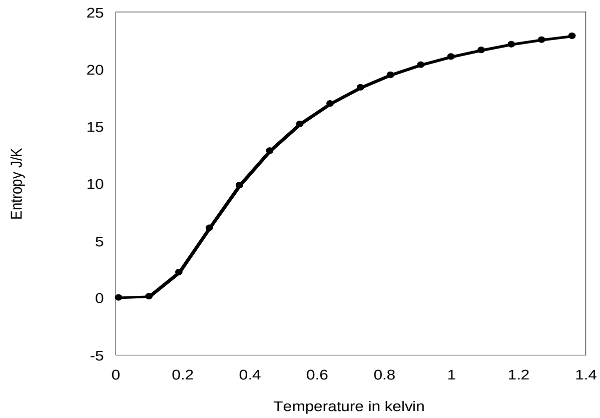

Figure 3: Variation of Entropy with temperature

Discussion

At low temperatures the main interactions are low energy pair interactions with more contribution from S-wave scattering and characterized, in this particular case, by scattering length. Of course we project that at absolute zero temperature (zero Kelvin temperature) particles form a condensate and fermions will appear to interact in such a manner as to cause the system to exhibit bosonic characteristics. The pairing of fermions will lead to zero total spin and momentum respectively, effectively forming a boson. The particles are paired in the sense that their movement is strongly correlated but not physically bound in close proximity like in a case of a molecule.

This in essence makes the boson-boson and fermion-fermion components to inherently possess nearly similar properties at near zero Kelvin temperature. The quantum degeneracy of the system is essentially the same but multiple occupation of a state is obviously forbidden for fermions.

The weak interaction of fermions is mediated by the lattice and approaches zero at absolute zero temperature Kelvin. Superfluidity in fermions is exhibited on the basis of interaction and is a phenomenon that is realized at any temperature below a finite transition temperature. The density distribution is expected to be similar because boson-boson interaction is repulsive as opposed to the quantum pressure for fermions. Bosons should be localized in the absence of any interactions with fermions whereas fermionic density fluctuations are localized within a specified distance-the localization length. The localization length of bosons is therefore controlled by the interaction with fermions. The BEC cannot be stable in a system that is basically attractive hence boson-boson particles will be expected to be destabilized close to a feshback resonance, unlike fermions which stabilizes.

The many body mixtures of particles with different quantum statistics are not well understood theoretically and are believed to show different behavior from pure systems of bosons and fermions. For low attraction between boson-fermion, there exist a Fermi sphere of fermions, the bosons will occupy the ground state and form a pure Bose Einstein condensate caused by purely bosonic quantum fluctuations. Increased boson-fermion causes abound state. BEC vanishes at a point when fermions are more than bosons as all bosons pair up with fermions. This is a second order phase transition which may occur at the peak of the curve on figure 2.

Particle disorder decreases with decrease in the total energy of the system. Figure 3 suggests that particles settles and interacts less as the system gives out energy. This is in agreement with conventional knowledge and concurs with what (Khanna *et al.*, 2010) determined when they studied the anharmonic perturbation of the neutron-proton pairs by the unpaired neutrons in heavily finite nuclei, where the emission of $\boldsymbol{\alpha}$ and γ radiations simply serves to stabilize the system. It's also in good agreement by works done by(Ayodo 2008, Ayodo *et al.*, 2010) on low temperature statistical thermodynamics of binary bose-fermi system , and(Chan *et al.,* 1992,1996) on the effect of disorder on the superfluid 3He-4He .

ACKNOWLEDGEMENT
We appreciate our Universities who have given us platforms to share our talents and skills as their dedicated employees.

Conclusion
Different authors (Molmer *et al.*, 1998), (Amoruso *et al.*, 1998), (Miyawaka *et al.*, 2000), (Nygaard *et al.*, 1999), (Roth *et al.,* 2002), (Akdeniz *et al.,* 2002), (Yi *et al.*, 2001), (Al-Hayek *et al.*, 1999) and (Mezzanti *et al.,* 2000) studied binary mixtures using different models. In this research, the particles are weakly interacting and assume that the bosons are in pure Bose-Eistein condensate, from which we conclude as follows; in the binary mixture of boson and fermion the energy density of bosons and fermions are independent of each other. The total energy of the binary system largely depends on the occupation number of particles, as the occupation number of particles increase energy increases in the same proportion. This concurs with works of other authors (Ayodo, 2008), (Khanna *et al.*, 2010), (Chan *et al.*, 1992, 1996).The values of specific heat for the binary mixture decreases as the mixture is cooled to near zero temperature Kelvin an indication that energy is being released to the surrounding as the system cools. The transition temperature for the ^{3}He-^{4}He mixture from figure 2 is 0.4*Kelvin.*Entropy is a measure of molecular disorder (Ayodo, *2008*) when the system cools, the internal energy of the particles decreases resulting in less and less particle motion. The graph in figure 3, confirms this observation by predicting that entropy decreases with decrease in temperature.

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