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Research Article

ON DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO W (L_P,ξ(T)) CLASS BY (N,P_N,Q_N)(E,1) PRODUCT MEANS OF ITS FOURIER SERIES

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ABSTRACT
In this paper, a new theorem on degree of approximation of a function belonging to the W (Lp, $\xi(t)$) class by (N, p_n, q_n) (E, 1) product summability means of Fourier series have been established.

Key Words: *Fourier Series, Degree of approximation, W $(Lp,\ \xi(t))$ class of fuction, (N, p_n, q_n) mean, (E, 1) mean, (N, p_n, q_n) (E,1) mean, Lebesgue Integral.*
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INTRODUCTION

The degree of approximation belonging to Lip α , Lip (α, p), Lip $(\xi(t), p)$ and W (Lp, $\xi(t)$) class using Cesároo, Nörlund and generalized Nörlund summability methods has been discussed a numbers of researchers like chandra (Borwein, 1958; Holland, 1981; Holland and Sahney, 1976; Khan, 1974; Lal and Tripathi, 2000; Qureshi1982 and Tiwari *et al.*, 2010) obtained a result on Degree of approximation of Weighted class functions by Product Means of its Fourier series and (Tiwari and Bhatt 2011) obtained a result on degree of approximation of the function belonging to Lip α Class by (N, p_n, q_n) (E, q) means of its Fourier series. In this paper our aim is to generalize the result of (Tiwari *et al.*, 2011) On Degree of approximation of weighted class functions by Product Means of its Fourier series.

Definition and Notation

Let f(x) be periodic with period 2π and integrable in the Legesgue sense. The Fourier series f(x) is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \qquad (2.1)$$

Lp - norm is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p},\ p \geq 1 \qquad (2.2)$$

L_∞- norm of a function f : R→R is defined by

$$\|f\|_\infty = Sup\{|f(x)| : x \in R\} \qquad (2.3)$$

The degree of approximation of a function of f : R→R by a trigonometric polynomial t_n of order n is defined by Zygmund (11)

$$\|t_n - f\|_\infty = Sup\{|t_n(x) - f(x)| :\ x \varepsilon R\} \qquad (2.4)$$

A function $f\ \varepsilon\ Lip\ \alpha$ if

$$|f(x+t)-f(x)| = O\left(|t|^{\alpha}\right), \text{ for } 0<\alpha\le 1 \tag{2.5}$$

A function f(x) ε Lip (α, p) for $0 \le x \le 2\pi$ if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^p\,dx\right)^{1/p} = O(|t|^{\alpha}),\ 0<\alpha\le 1, p\ge 1 \tag{2.6}$$

Given a positive increasing function $\xi(t)$ and an integer $p \ge 1$, $f\ \varepsilon$ Lip $(\xi(t), p)$, *if*

$$\left(\int_0^{2\pi} |\{f(x+t)-f(x)\}|^p\,dx\right)^{1/p} = O(\xi(t)) \tag{2.7}$$

and $f\ \varepsilon W(Lp, \xi(t))$ *if*

$$\left(\int_0^{2\pi} \left|\{f(x+t)-f(x)\}\sin^{\beta} x\right|^p\,dx\right)^{1/p} = O(\xi(t)),\ \beta \ge 0 \tag{2.8}$$

In case $\beta = 0$, W $(Lp, \xi(t))$ class reduces to the Lip $(\xi(t), p)$ class and if $\xi(t) = t^{\alpha}$, then Lip $(\xi(t), p)$ class reduces to the Lip (α, p) class and if $p \to \infty$ then Lip (α, p) class reduces to the Lip α class. We observe that

$$Lip\ \alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(Lp, \xi(t))\ for\ 0<\alpha\le 1,\ \ p\ge 1$$

Let $\sum_{n=1}^{\infty} u_n$ be a given infinite series with sequence of its n[th] partial sums $\{s_n\}$.
The (E,1) transform is defined as the n[th] partial sum of (E,1) summability, its denoted by E_n^1 and given by

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k \to s, \text{ as } n \to \infty \tag{2.9}$$

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive constants such that

$$P_n = \sum_{k=0}^{n} p_k$$

$$Q_n = \sum_{k=0}^{n} q_k$$

$$\text{and } R_n = \sum_{k=0}^{n} p_k\, q_{n-k} \ne 0\ (n \ge 0)$$

Where P_n, Q_n and $R_n \to \infty$ as $n \to \infty$

For two given sequences $\{p_n\}$ and $\{q_n\}$ the convolution $(p*q)_n$ is defined by

$$R_n = (p*q)_n = \sum_{k=0}^{n} p_k\, q_{n-k}$$

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\, s_k \quad .$$

The generalized Nörlund transform (N,p_n,q_n) of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to s$ as $n\to\infty$ then the sequence $\{s_n\}$is said to be summable by generalized Nörlund method (N,p_n,q_n) to s (Borwein(1)).

The necessary and sufficient condition for (N, p_n, q_n) method of summability to be regular are

$\sum_{k=0}^{n}|p_{n-k}\, q_k| = O(|R_n|)$ and $p_{n-k} = O(|R_n|)$, as $n\to\infty$, for every fixed $k\geq 0$,for which $q_k\neq 0$.

The (N,p_n,q_n) transform of the (E,1) transform is defines as

$$(N, p_n, q_n)(E,1) = \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\, E_{n-k}^{1} \tag{2.10}$$

If

$$(N, p_n, q_n)(E,1) = \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\, E_{n-k}^{1} \to s \; as \quad n\to\infty$$

Then $\sum_{n=1}^{\infty} u_n$ is said to be (N, p_n, q_n) (E,1) summable to s.

We shall use the following notations:-

(i) $\phi(t) = f(x+t) + f(x-t) - 2f(x)$ (2.11)

(ii) $N_n^{p,q,E_1}(t) = \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k} \cos^{(n-k)}\frac{t}{2}.\sin(n-k+1)\frac{t}{2}$ (2.12)

Known Reults

Tiwari et al.(10) obtained a result On Degree of approximation of Weighted class functions by Product Means of its Fourier series.They have proved the following theorem:

Theorem 3.1: If $f : R \to R$ is 2π- periodic function belonging to the Weighted class $W(L^p,\xi(t)),(p\geq 1)$ then the degree of approximation of function f by the (N,p_n) (E,1) product means of its Fourier series satisfies ,for n=0,1,2,3,…

$$\|(N, p_n)(E,1) - f(x)\|_p = O\left[(n+1)^{\beta+\frac{1}{p}}\; \xi\left(\frac{1}{n+1}\right)\right]$$

Provided $\{p_n\}$ is non-negative, monotonic and non-increasing sequence and $\xi(t)$ satisfies the following conditions

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}Sin^{\beta}t\right)^p dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right) \tag{3.1}$$

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,Sin^{\beta}t}{\xi(t)}|\phi(t)|\right)^p dt\right\}^{1/p} = O\left((n+1)^{\delta}\right) \tag{3.2}$$

Where δ is an arbitrary number such that q (1-δ)-1 >0,

Conditions (3.1) and (3.2) holds uniformly in x and

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\left(\frac{\xi(t)}{t^{1+\beta}}\right)^q\right)dt\right\}^{1/q} = O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{p}}\right)$$

Also hold in limit $t\varepsilon\left[\frac{\pi}{n+1},\pi\right]$,where $\frac{1}{p}+\frac{1}{q}=1$ Such that $1\leq p\leq\infty$.

***Main Theorem*:** In this paper our aim is to generalize the above result of Tiwari *et al*., 2010. In fact, we prove the following theorem:

Let (N,p_n,q_n) be a regular generalized Nörlund method defined by a positive,monotonic,non-increasing sequences {p_n} and {q_n}of real constants such that $R_n=\sum_{k=0}^{n}p_k q_{n-k}\neq 0.$. If f is 2π- periodic function, Lebesgue integrable on (-π,π) and is belonging to Weighted W(L_p,ξ(t)) class,p≥1,then the degree of approximation of function f is given by

$$\|(N,p_n,q_n)(E,1)-f(x)\|_p = O\left[(n+1)^{\beta+\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right] \tag{4.1}$$

Provided □ (t) satisfies the folowing conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is a decreasing sequence ,} \tag{4.2}$$

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p}t\,dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right) \tag{4.3}$$

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left((n+1)^{\delta}\right) \tag{4.4}$$

Where δ is an arbitrary number such that q (1-δ)-1 >0,

Conditions (4.3) and (4.4) holds uniformly in x and where $\frac{1}{p}+\frac{1}{q}=1$ Such that $1\leq p\leq\infty$.

***Lemmas*:** For the proof of our theorem, we require following lemmas:

Lemma (5.1): $\left|N_n^{p,q,E_1}(t)\right| = O(n+1)t, \quad for\ o \le t \le \frac{\pi}{n+1}.$

Lemma (5.2): $\left|N_n^{p,q,E_1}(t)\right| = O(1), \quad for\ \frac{\pi}{n+1} \le t \le \pi.$

Proof: (Lemma (5.1):

For $0 \le t \le \frac{\pi}{n+1};\ we\, have$

$$\left|N_n^{p,q,E_1}(t)\right| = \left|\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k} Cos^{(n-k)}\frac{t}{2}.Sin\,(n-k+1)\frac{t}{2}\right|$$

$$\le \frac{1}{R_n}\sum_{k=0}^{n} p_n\, q_{n-k}\left|Cos^{(n-k)}\frac{t}{2}\right|.\left|Sin(n-k+1)\frac{t}{2}\right|$$

$$=\frac{1}{R_n}\sum_{k=0}^{n} p_n.q_{n-k}.1.(n-k+1)\frac{t}{2} \quad (\because Sin(n-k+1)\frac{t}{2} \le (n-k+1)\frac{t}{2} and\ Cos^{(n-k)}\frac{t}{2} \le 1)$$

$$= O\big((n+1)t\big)\left\{\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\right\}$$

$$\left|N_n^{p,q,E_1}(t)\right| = O\big((n+1)t\big).$$

This complete proof of the lemma (5.1).

Proof: (Lemma (5.2):

For $\frac{\pi}{n+1} \le t \le \pi; we\ have$

$$\left|N_n^{p,q,E_1}(t)\right| = \left|\frac{1}{R_n}\sum_{k=0}^{n} p_n\, q_{n-k}\, Cos^{(n-k)}\frac{t}{2} Sin(n-k+1)\frac{t}{2}\right|$$

$$\le \frac{1}{R_n}\sum_{k=0}^{n} p_n\, q_{n-k}\left|Cos^{(n-k)}\frac{t}{2}\right|.\left|Sin(n-k+1)\frac{t}{2}\right|$$

$$\le \frac{1}{Rn}\sum_{k=0}^{n} p_n\, q_{n-k}.1.1 \quad (Because\ \ Cos^{(n-k)}\frac{t}{2} \le 1)$$

$$= O(1)\left\{\frac{1}{R_n}\sum_{k=0}^{n} p_k \quad q_{n-k}\right\}$$

$$\left|N_n^{p,q,E_1}(t)\right| = O(1).$$

This complete proof of the lemma (5.2)

***Proof of the Main Theorem*:**
Let $S_n(x)$ denotes the n^{th} partial sum of the Fourier series at t=x we have

$$E^1_{n-k}(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\phi(t)}{Sin\, t/2} Cos^{(n-k)}\frac{t}{2}.Sin\,(n-k+1)\frac{t}{2}\,dt$$

Now,

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\left\{E^1_{n-k}(x) - f(x)\right\} = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{Sin\,t/2}.\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\, Cos^{(n-k)}\, t/2.Sin(n-k+1)\frac{t}{2}\,dt$$

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\, E^1_{n-k}(x) - \frac{1}{R_n}\sum_{k=0}^{n} p_n\, q_{n-k}\, f(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{Sin\,t/2} N_n^{p,q,E_1}(t)\,dt$$

(By(2.12))

$$(N,p_n,q_n)(E,1)\text{-}f(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{Sin\, t/2} N_n^{p,q,E_1}(t)\,dt$$

(By(2.10))

Since, $Sin\,t/2 \geq \frac{t}{\pi}$, *it follows that*

$$\left|(N,p_n,q_n)(E,1) - f(x)\right| \leq \frac{1}{2}\int_0^{\pi}\frac{|\phi(t)|}{t}\left|N_n^{p,q,E_1}(t)\right|dt$$

$$= \frac{1}{2}\left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right)\frac{|\phi(t)|}{t}\left|N_n^{p,q,E_1}(t)\right|dt$$

$$= \frac{1}{2}\int_0^{\pi/n+1}\frac{|\phi(t)|}{t}\left|N_n^{p,q,E_1}(t)\right|dt + \frac{1}{2}\int_{\pi/n+1}^{\pi}\frac{|\phi(t)|}{t}\left|N_n^{p,q,E_1}(t)\right|dt$$

$$= I_1 + I_2\ (Say) \qquad (6.1)$$

Let us consider, I_1 first

$$I_1 = \frac{1}{2}\int_0^{\pi/n+1}\frac{|\phi(t)|}{t}\left|N_n^{p,q,E_1}(t)\right|dt$$

Now using Hölder inequality and in view of $(\sin t)^{-1} \leq \frac{\pi}{t}, 0 < t \leq \frac{\pi}{2}$

$$I_1 = \frac{1}{2t}\left\{\int_0^{\pi/n+1}\left[\left[\frac{t|\phi(t)|}{\xi(t)}\right] Sin^{\beta}t\right]^p dt\right\}^{1/p}.\left\{\int_0^{\pi/n+1}\left[\frac{\xi(t)}{t\,Sin^{\beta}t}.\left|N_n^{p,q,E_1}(t)\right|\right]^q dt\right\}^{1/q}$$

$$= O\left(\frac{1}{n+1}\right)O((n+1)t).\frac{1}{2t}\left\{\int_0^{\pi/n+1}\left[\frac{\xi(t)}{t^{1+\beta}}\right]^q dt\right\}^{1/q} \qquad (By.\ (4.3)\ and\ Lemma(5.1))$$

$$=O(1)\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{p}}\right)$$

$$I_1 = O\left((n+1)^{\beta+\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right) \qquad (6.2)$$

Now consider $I_2 = \frac{1}{2}\int_{\pi/n+1}^{\pi} \frac{|\phi(t)|}{t}.\left|N_n^{p,q,E_1}(t)\right|dt$

Again using Hölder ineqality and in view of $\text{Sin}\,\theta \geq \frac{2t}{\pi}$

$$I_2 = \frac{1}{2t}\left\{\int_{\pi/n+1}^{\pi}\left(\frac{t^{-\delta}Sin^{\beta}t}{\xi(t)}|\phi(t)|\right)^p dt\right\}^{1/p}.\left\{\int_{\pi/n+1}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}Sin^{\beta}t}.\left|N_n^{p,q,E_1}\right|\right)^q dt\right\}^{1/q}$$

$$= O\left((n+1)^{\delta}\right)O(1).\frac{1}{2t}\left\{\int_{\pi/n+1}^{\pi}\left[\frac{\xi(t)}{t^{-\delta}Sin^{\beta}t}\right]^q dt\right\}^{1/q} \quad (by.(4.4)\ and Lemma(5.2)$$

$$= O\left((n+1)^{\delta}\right).\frac{1}{2}\left\{\int_{\pi/n+1}^{\pi}\left(\frac{\xi(t)}{t^{1-\delta}Sin^{\beta}t}\right)^q dt\right\}^{1/q}$$

$$=O((n+1)^{\delta})\,O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{p}-\delta}\right)$$

$$I_2 = O\left((n+1)^{\beta+\frac{1}{p}}\ \xi\left(\frac{1}{n+1}\right)\right) \qquad (6.3)$$

Combining equation (6.1) (6.2)and(6.3),we have

$$\left|(N, p_n, q_n)(E,1) - f(x)\right| = O\left((n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right)$$

Using the Lp-norm we get

$$\left\|(N, p_n, q_n)(E,1) - f(x)\right\|_p = \left(\int_0^{2\pi}\left|(N, p_n, q_n)(E,1) - f(x)\right|^p dx\right)^{1/p}, (1 \leq p < \infty)$$

$$= O\left[\left\{\int_0^{2\pi}(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)^p dx\right\}^{1/p}\right]$$

$$= O\left[(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right], 1 \leq p < \infty$$

This completes the proof of the our theorem.

***Applications*:** The following corollaries can be derived from the theorem:
Corollary 7.1: If β=0 and ξ(t) = t^{α} , 0 <α ≤ 1 , then the W (L_p, t) class, $p \geq 1$,reduces to Lip(α,p) class and the degree of approximation of a function $f(x)$, 2π- periodic function f ε Lip(α,p) , $\frac{1}{p}$ < α ≤ 1 is given by

$$\left\|(N, p_n, q_n)(E,1) - f(x)\right\|_p = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}}\right).$$

Corollary7.2: If p→∞ in corollary 7.1 , for 0 < α < 1 ,then the Lip(α,p) class reduces to Lip α class and the degree of approximation of a function $f(x)$, 2π- periodic function f ε Lip α , 0 < α < 1 is given by

$$\left\|(N, p_n, q_n)(E,1) - f(x)\right\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}}\right).$$

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