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Research Article

STEINHAUS TYPE THEOREMS FOR SUMMABILITY MATRICES IN ULTRAMETRIC FIELDS

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ABSTRACT
Throughout this note, K denotes a complete, non-trivially valued, ultrametric field of characteristic zero. Infinite matrices and sequences have entries in K. We prove Steinhaus type theorems for the Nörlund, Natarajan and Euler summability matrices in K.
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INTRODUCTION
In this short note, K denotes a complete, non-trivially valued, ultrametric field of characteristic zero. The p-adic field Q_p for a prime p is one such field. Infinite matrices and sequences have entries in K. To make the paper self-contained, we recall certain concepts, Definitions and Theorems. Given an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, n, k = 0, 1, 2, ... and a sequence $x = \{x_k\}$, $x_k \in K$, k = 0, 1, 2, ..., by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is summable A or A-summable to ℓ.

If X, Y are sequence spaces in K, we write $A = (a_{nk}) \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. Let ℓ_∞ denote the ultrametric Banach space of all bounded sequences in K and c denote the closed subspace of ℓ_∞ consisting of all convergent sequences in K. If $A \in (c, c)$, we say that A is "conservative". If $A \in (c, c)$ and $\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k$, $x = \{x_k\} \in c$, we say that A is regular. The set of all regular matrices is denoted by (c, c; P), P denoting "preservation of limits". The following result, which characterizes a conservative matrix and a regular matrix in terms of its entries, is well-known (see, for instance, (Monna, 1963)).

Theorem 1.1: $A = (a_{nk}) \in (c, c)$, i.e., A is conservative if and only if

$$\sup_{n,k} |a_{nk}| < \infty; \tag{1.1}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \quad k = 0,1,2,...; \tag{1.2}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha. \tag{1.3}$$

Further, $A \in (c, c; P)$, i.e., A is regular if and only if (1.1), (1.2), (1.3) hold with $\alpha_k = 0$, k = 0, 1, 2, ... and $\alpha = 1$.

Again, the following result is well-known (Natarajan, 1978).

Theorem 1.2: (Steinhaus) Given any regular matrix A, there is a bounded, divergent sequence which is not A-summable. Symbolically, the result can be written as
$(c, c; P) \cap (\ell_\infty, c) = \phi$.

We call any result of the above form a "Steinhaus type theorem".

Steinhaus Type Theorems in K

If X, Y are sequences spaces with a notion of "limit", by (X, Y; P), we mean the set of all infinite matrices $A \in (X, Y)$ with the "preservation of corresponding limits".

Steinhaus type theorems in K were studied earlier by Natarajan in 1987, 1996, 1999, 2008.

We recall the following which is needed in the sequel.

Definition 2.1: (Srinivasan,1965) Given the sequence $p = \{p_n\}$, the Nörlund method (N, p_n) is defined by the infinite matrix (a_{nk}), where

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & k \le n; \\ 0, & k > n, \end{cases}$$

where $|p_n| < |p_0|$, n = 1, 2, ... and $P_n = \sum_{k=0}^{n} p_k$, n = 0, 1, 2, ... This matrix (a_{nk}) is denoted by (N, p) and it is called a Nörlund matrix.

Theorem 2.2: (Natarajan, 1994) The Nörlund method (N, p_n) is regular if and only if
$\lim_{n\to\infty} p_n = 0.$

Definition 2.3: (Natarajan, 2003) Given $\lambda = \{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = 0,$ the Natarajan method (M, λ_n) is defined by the infinite matrix (b_{nk}), where

$$b_{nk} = \begin{cases} \lambda_{n-k}, & k \le n; \\ 0, & k > n. \end{cases}$$

The matrix (b_{nk}) is denoted by (M, λ) and we call it a Natarajan matrix.

Theorem 2.4: (Natarajan, 2012) The Natarajan method (M, λ_n) is regular if and only if
$\sum_{n=0}^{\infty} \lambda_n = 1.$

Definition 2.5: (Natarajan, 2003) Let $r \in K$ such that $|1-r| < 1$. Then Euler method of order r or the (E, r) method is defined by the infinite matrix $(e_{nk}^{(r)})$, where:
If $r \ne 1$,

$$e_{nk}^{(r)} = \begin{cases} {}^nC_k r^k (1-r)^{n-k}, & k \le n; \\ 0, & k > n, \end{cases}$$

$${}^nC_k = \frac{n!}{k!(n-k)!}, \ k \le n;$$

If r = 1,

$$e_{nk}^{(1)} = \begin{cases} 1, & k = n; \\ 0, & k \ne n. \end{cases}$$

$(e_{nk}^{(r)})$ is called the (E, r) matrix or Euler matrix of order r.

Theorem 2.6: (Natarajan, 2003) The (E, r) method is always regular.

We now prove Steinhaus type theorems for the Nörlund, Natarajan and Euler matrices.

For convenience, we denote the set of all sequences which are (N, p_n), (M, λ_n), (E, r) summable by (N, p), (M,λ), (E, r) respectively (so that we understand the meaning of these symbols according to the context). In this context, we note that (N, p), (M, λ), (E, r) $\subseteq \ell_\infty$ (see Natarajan, 2012; Deepa *et al.*, Srinivasan, 1965).

Theorem 2.7: A = $(a_{nk}) \in$ (c, (N, p)) if and only if

$$\lim_{k\to\infty} a_{nk} = 0, \quad n = 0,1,2,...; \tag{2.1}$$

$$\sup_{n,k} \left| \frac{1}{P_n} \sum_{i=0}^{n} p_i a_{n-i,k} \right| < \infty; \tag{2.2}$$

$$\lim_{n\to\infty} \left(\frac{1}{P_n} \sum_{i=0}^{n} p_i a_{n-i,k} \right) = \beta_k, \quad k = 0,1,2,...; \tag{2.3}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \left(\frac{1}{P_n} \sum_{i=0}^{n} p_i a_{n-i,k} \right) = \beta. \tag{2.4}$$

Proof. Sufficiency. Let (2.1), (2.2), (2.3), (2.4) hold. For x = $\{x_k\} \in$ c, in view of (2.1),

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...$$

is defined. Let B = (b_{nk}), $b_{nk} = \frac{1}{P_n} \sum_{i=0}^{n} p_i a_{n-i,k}$, n, k = 0, 1, 2, ...

Using (2.2), (2.3) and (2.4), B $\in$ (c, c) in view of Theorem 1.1.

Now,

$$\left\{ \sum_{k=0}^{\infty} b_{nk} x_k \right\}_{n=0}^{\infty} \in c,$$

i.e., $\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n}p_i a_{n-i,k}\right)x_k\right\}_{n=0}^{\infty} \in c,$

i.e., $\left\{\frac{1}{P_n}\sum_{i=0}^{n}p_i\left(\sum_{k=0}^{\infty}a_{n-i,k}x_k\right)\right\}_{n=0}^{\infty} \in c,$

i.e., $\left\{\frac{1}{P_n}\sum_{i=0}^{n}p_i(Ax)_{n-i}\right\}_{n=0}^{\infty} \in c,$

i.e., $\{(Ax)_n\}_{n=0}^{\infty} \in (N,p).$

Thus $A \in (c, (N, p))$.

Necessity. Let $A \in (c, (N, p))$. So, for $x = \{x_k\} \in c$, $\{(Ax)_n\} \in (N, p)$.
Retracing the above steps, it is clear that $B \in (c, c)$. Appealing to Theorem 1.1, (2.2), (2.3), (2.4) hold. Considering the A-transform of the convergent sequence {1, 1, 1, ...}, we note that $\sum_{k=0}^{\infty} a_{nk}$ converges, n = 0, 1, 2, ... so that (2.1) holds. This completes the proof of the theorem.

Corollary 2.8: $A \in (c, (N, p); P)$, i.e., $A \in (c, (N, p))$ with
$\lim_{n\to\infty}\frac{1}{P_n}\{p_0(Ax)_n + p_1(Ax)_{n-1} + \cdots + p_n(Ax)_0\} = \lim_{k\to\infty} x_k$, $x = \{x_k\} \in c$, if and only if (2.1), (2.2), (2.3), (2.4) hold with $\beta_k = 0$, k = 0, 1, 2, ... and $\beta = 1$.

We can prove the following theorem on similar lines.

Theorem 2.9: $A = (a_{nk}) \in (\ell_\infty, (N, p))$ if and only if (2.1), (2.3) hold and

$$\lim_{n\to\infty}\sup_{k\geq 0}\left|\sum_{i=0}^{n+1}p_i(a_{n+1-i,k} - a_{n-i,k})\right| = 0, \tag{2.5}$$

where we define $a_{nk} = 0$ when $n < 0$ or $k < 0$.

We now deduce the following Steinhaus type theorem.

Theorem 2.10: $(c, (N, p); P) \cap (\ell_\infty, (N, p)) = \phi$.

Proof. Let $A = (a_{nk}) \in (c, (N, p); P) \cap (\ell_\infty, (N, p))$.
In view of (2.5), using the fact that $|P_n| = |p_0|$, n = 0, 1, 2, ...,

$\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n}p_i a_{n-i,k}\right)$ converges uniformly in n

and so

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n}p_i a_{n-i,k}\right) = \sum_{k=0}^{\infty}\lim_{n\to\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n}p_i a_{n-i,k}\right) = 0,$$

which is a contradiction, using Corollary 2.8. This completes the proof.

The proofs of the following theorems are similar.

Theorem 2.11. A = $(a_{nk}) \in (c, (M, \lambda))$ if and only if (2.1) holds,

$$\sup_{n,k\ge 0}\left|\sum_{i=0}^{n}\lambda_i a_{n-i,k}\right| < \infty; \tag{2.6}$$

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n}\lambda_i a_{n-i,k}\right) = \gamma_k, \quad k = 0,1,2,...; \tag{2.7}$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{n}\lambda_i a_{n-i,k}\right) = \gamma. \tag{2.8}$$

Corollary 2.12: A $\in$ (c, (M,λ); P) if and only (2.1), (2.6), (2.7), (2.8) hold with $\gamma_k = 0$, k = 0 1, 2, ... and $\gamma = 1$.

Theorem 2.13: A = $(a_{nk}) \in (\ell_\infty, (M,\lambda))$ if and only if (2.1), (2.7) hold and (2.5) holds with p_i replaced by λ_i.

Theorem 2.14: (Steinhaus type) (c, (M, λ); P) $\cap$ (ℓ_∞, (M, λ)) = ϕ.

Theorem 2.15: A = $(a_{nk}) \in (c, (E, r))$ if and only if (2.1) holds,

$$\sup_{n,k\ge 0}\left|\sum_{j=0}^{n} {}^{n}C_j r^j (1-r)^{n-j} a_{jk}\right| < \infty; \tag{2.9}$$

$$\lim_{n\to\infty}\left(\sum_{j=0}^{n} {}^{n}C_j r^j (1-r)^{n-j} a_{jk}\right) = \delta_k, \quad k = 0,1,2,...; \tag{2.10}$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{n} {}^{n}C_j r^j (1-r)^{n-j} a_{jk}\right) = \delta. \tag{2.11}$$

Corollary 2.16: A $\in$ (c, (E, r); P) if and only if (2.1), (2.9), (2.10), (2.11) hold with $\delta_k = 0$, k = 0, 1, 2, ... and $\delta = 1$.

Theorem 2.17: A = $(a_{nk}) \in (\ell_\infty, (E, r))$ if and only if (2.1), (2.10) hold and

$$\lim_{n\to\infty}\sup_{k\ge 0}\left|\sum_{j=0}^{n+1} {}^{(n+1)}C_j r^j (1-r)^{n+1-j} a_{jk} - \sum_{j=0}^{n} {}^{n}C_j r^j (1-r)^{n-j} a_{jk}\right| = 0. \tag{2.12}$$

Theorem 2.18: (Steinhaus type) (c, (E, r); P) $\cap$ (ℓ_∞, (E, r)) = ϕ.

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