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Research Article

COMMON FIXED POINT THEOREM IN FUZZY NORMED SPACES

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ABSTRACT
Chugh and Rathi (Chugh and Rathi, 2005) introduced the concept of Fuzzy normed space The aim of this paper is to prove a common fixed point theorem for a pair of operators in fuzzy normed space.

Key Word: *Fixed Point, Fuzzy Normed Space*

INTRODUCTION

In 1942, K. Menger introduced the notion of PM-space by generalizing the concept of metric space to the situation when we do not know the distance between the points. It is suitable to look upon the distance concept as a statistical or probabilistic rather than deterministic one, because the advantage of a probabilistic approach is that it permits from the initial formulation a greater flexibility rather than that offered by a deterministic approach.

The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus, for any p, q elements in the space, we have a distribution function F(p, q; x) and interpret F(p, q; x) as the probability that distance between p and q is less than x.

In 1963, Serstnev generalized the concept of ordinary normed space to random normed space. In fact, a random normed space is a menger space if we set $G_{x,y} = F_{x-y}$. Fixed point theorems for contraction mappings in RN-spaces were first investigated by Boscan (1974). Thus many fixed point theorems for metric space have an immediate analogue in Random normed spaces. For topological preliminaries in RN – spaces, Schweizer and Sklar (1983) and Serstnev (1963) are excellent readings.

Zadeh, 1965 introduced the concept of fuzzy set. Since then, a lot of work has been developed by many authors regarding the theory of fuzzy sets and applications. Especially, Erceg, (1979); Kaleva and Sekkala (1984); Kramosil and Michalek (1975) have introduced the concept of fuzzy metric space in different ways. Grabiec, 1983 followed Kramosil and Michalek (1975) and obtained the fuzzy version of Banach contraction principle. Grabiec (1983) results were further generalized by Subrahmanyam, 1995 for a pair of commuting mappings. Moreover, George and Veeramani (1994) modified the concept of fuzzy metric space, introduced by Kramosil and Michalek (1975) and introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric induces a fuzzy metric. Chugh and Rathi (2005) introduced the concept of Fuzzy normed spaces. In this paper, we prove a common fixed point theorem for a pair of operators in fuzzy normed space.

A fuzzy normed space is a fuzzy metric space if $G(x, y, t) = M(x-y, t)$.

Definition 1.1 A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2 A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if * satisfies the following conditions :

1. * is associative and commutative,
2. * is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \le c * d$ whenever $a \le c$ and $b \le d$, $(a, b, c, d \in [0, 1])$.

Example 1.1 $a*b = ab$, $a*b = \min\{a, b\}$.

Definition 1.3 A triplet (X, M, *) is called a fuzzy normed space (briefly FN – space) if X is a real vector space, * is a continuous t-norm and M is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions

(FN – 1) $M(x, 0) = 0$,

(FN – 2) $M(x, t) = 1$ for all $t > 0$ if and only if $x = 0$, (FN–3) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ for all $\alpha \in R$, $\alpha \neq 0$,

(FN–4) $M(x + y, t + s) \geq M(x, t) * M(y, s)$ for all $x, y \in X$ and $t, s \in R+$

(FN–5) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous for all $x \in X$

(FN–6) $\lim_{t\to\infty} M(x, t) = 1$ for all x in X and $t \in R+$.

A fuzzy normed space is a fuzzy metric space if we set $G(x, y, t) = M(x - y, t)$.

Remark 1 M(x, t) can be thought of as the degree of nearness of norm of x with respect to t.

Definition 1.4 The natural topology t(M) is said to be topological if for each x in X and any $\in > 0$

$U_x(\in) = \{y : M(x-y, \in) > 1-\in\}$ is a neighbourhood of x in t(M).

Definition 1.5 A sequence $\{x_n\}$ in a fuzzy normed space is said to be convergent if for each r, $0 < r < 1$, and $t > 0$, there exists $n_0 \in N$ such that

$M(x_n-x, t) > 1-r$ for all $n \geq n_0$.

Definition 1.6 A sequence $\{x_n\}$ in a fuzzy normed space is said to be a Cauchy if for each r, $0 < r < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$M(x_n- x_m, t) > 1-r$ for all $n, m \geq n_0$.

Definition 1.7 A fuzzy normed space is said to be complete if every Cauchy sequence is convergent.

Example 1.2 Let X = R. Define $a * b = ab$ and $M(x, t) = \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1}$ for all $x \in X$ and $t \in [0, \infty)$.

Then (X, M, *) is a fuzzy normed space.

Proof (i) $M(x, 0) = 0$

(ii) $M(x, t) = 1$ implies $\left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} = 1$

or $\exp\left(\frac{|x|}{t}\right) = 1$

or $\exp\left(\frac{|x|}{t}\right) = \exp(0)$

or $\left(\frac{|x|}{t}\right) = 0$

$\Rightarrow$ $|x| = 0 \Rightarrow x = 0$.

If $x = 0$, then $M(x, t) = 1$.

(iii) $M(\alpha x, t) = M(x, \frac{t}{|\alpha|})$ is obvious

(iv) To prove

$M(x+y, t+s) \geq M(x, t) * M(y, s)$

Since $\frac{|x+y|}{t+s} \leq \frac{|x|+|y|}{t+s}$

$= \frac{|x|}{t+s} + \frac{|y|}{t+s}$

$$\leq \frac{|x|}{t} + \frac{|y|}{s}$$

$$\exp\left(\frac{|x+y|}{t+s}\right) \leq \exp\left(\frac{|x|}{t} + \frac{|y|}{s}\right)$$

$$= \exp\left(\frac{|x|}{t}\right).\exp\left(\frac{|y|}{s}\right)$$

Taking inverse, we have

$$\left[\exp\left(\frac{|x+y|}{t+s}\right)\right]^{-1} \geq \left[\exp\left(\frac{|x|}{t}\right)\right]^{-1} . \left[\exp\left(\frac{|y|}{s}\right)\right]^{-1}$$

Hence $M(x+y, t+s) \geq M(x, t) * M(y, s)$

(v) $M(x, .) : [0, \infty) \to [0, 1]$ is left continuous

(vi) $\lim_{t\to\infty} M(x,t) = 1$

Hence (X, M, *) is a fuzzy normed space.

Example 1.3: Let M be a fuzzy set on $X \times [0, \infty)$ defined by $M(x, t) = \frac{t}{t+|x|}$ for all $x \in X, t > 0$ and * is a t-norm defined by $a*b = ab$. Then (X, M, *) is a fuzzy normed space.

Throughout this paper, we refer to X as a complete fuzzy normed space and Y a complete fuzzy normed space consisting of all continuous functions f from a finite closed interval I to X such that

$$M(f, x)_Y = \inf_{k\in I} M(f(k), x)_X, \text{ for every f in Y.}$$

Here $M(f, x)_Y$ and $M(f, x)_X$ stand for the degree of nearness of norm of f w.r.t. x in Y and X respectively. Moreover, the t-norm * in both X and Y is supposed to satisfy $*(x, y) = \min(x, y)$, $x, y \in [0, 1]$.

Definition (1.8): Let P be an operator from Y to X. An element $f \in Y$ is said to be a fixed point of P if $Pf = f(k)$ for some k in I.

Lemma 1.1: Let f_n be a sequence in Y, * is continuous t-norm and satisfies $*(t, t) \geq t$ for every $t \in [0, 1]$. If there exists a constant $q \in (0, 1)$ such that

$$M(f_n - f_{n+1}, qt)_Y \geq M(f_{n-1} - f_n, t)_Y \quad \ldots(1.1)$$

for all n, then $\{y_n\}$ is a Cauchy sequence.

Proof: Let $\in$, r be positive reals. Then for $m \geq n$ we have by (FN – 4),

$$M(f_n - f_m, \in)_Y \geq M(f_n - f_{n+1}, \in - q\in)_Y * M(f_{n+1} - f_m, q\in)_Y$$
$$\geq M(f_0 - f_1, (\in - q\in)\, q^{-n})_Y * M(f_{n+1} - f_m, q\in)_Y, \text{ by (1.1)}$$

Taking $(\in - q\in)q^{-n} = h$, we get

$$M(f_n - f_m, \in) \geq M(f_0 - f_1, h)_Y * (M(f_{n+1} - f_{n+2}, q\in - q^2\in)_Y * M(f_{n+2} - f_m, q^2\in)_Y)$$
$$\geq M(f_0 - f_1, h)_Y * (M(f_0 - f_1, h)_Y * M(f_{n+2} - f_m, q^2\in)_Y)$$

Since t-norm is associative and $*(t, t) \geq t$, we have

$$M(f_n - f_m, \in)_Y \geq M(f_0 - f_1, h)_Y * M(f_{n+2} - f_m, q^2\in)_Y$$

Repeating these arguments

$$M(f_n - f_m, \in)_Y \geq M(f_0 - f_1, h)_Y * M(f_{m-1} - f_m, q^{m-n-1}\in)_Y$$
$$\geq M(f_0 - f_1, h)_Y * M(f_0 - f_1, q^{-n}\in)_Y$$
$$\geq M(f_0 - f_1, h)_Y * M(f_0 - f_1, h)_Y$$

$$\geq M(f_0 - f_1, (\in - q\in)q^{-n})_Y$$

Therefore, if N be so chosen that $M(f_0 - f_1, (\in - q\in)\, q^{-N}) > 1 - r$, it follows that

$$M(f_n - f_m, \in) > 1 - r \text{ for all } m > n \geq N.$$

Hence $\{y_n\}$ is a Cauchy sequence.

Main Theorem

Theorem: Let P, Q be the operators from Y to X satisfying

(1) $$M(Pf - Qg, (qx))_X \geq \min\{M(f - g, x)_Y, M(f(k) - Pf, x)_X, M(g(k) - Pf, 2x)_X, M(f(k) - Qg, 2x)_X\}$$

for all $f, g \in Y$, $x \geq 0$, $q \in (0, 1)$ and $k \in I$. Then

(A) Given $f_0 \in Y$, every sequence $\{f_n\}$ satisfying

$$Pf_{2n} = f_{2n+1}(k),\ Qf_{2n+1} = f_{2n+2}(k), \quad n = 0, 1, 2 \ldots$$

and $$M(f_{n+1} - f_n, x)_Y = M(f_{n+1}(k) - f_n(k), x)_X$$

converges to a common fixed point f* of P and Q.

(2) Let $\{f_n\}$ and $\{g_n\}$ be the sequences of iterates of $f_0, g_0 \in Y$ constructed as in (A). If

$$S = \{f \in Y: M(f, x)_Y = M(f(k), x)_X\}$$

and $f_n - g_n \in S$ for every n, then

(B) $\lim f_n = \lim g_n$

(3) if f* is a common fixed point of P and Q and

$$S_{f^*} = \{f \in Y: M(f - f^*, x)_Y = M(f(k) - f^*(k), x)_X,$$

Then

(C) f* is the unique common fixed point of P and Q in S_f*.

Proof: for $f_0 \in Y$, constructing $\{f_n\}$ as in (A) and using (1)

$$\begin{aligned} M(f_{2n+1} - f_{2n+2}, qx)_Y &= M(f_{2n+1}(k) - f_{2n+2}(k), qx)_X \\ &= M(Pf_{2n} - Qf_{2n+1}, qx)_X \\ &\geq \min\{M(f_{2n} - f_{2n+1}, x)_Y, M(f_{2n}(k) - f_{2n+1}(k), x)_X, \\ &\quad M(f_{2n+1}(k) - f_{2n+2}(k), x)_X, M(f_{2n+1}(k) - f_{2n+1}(k), 2x)_X, \\ &\quad M(f_{2n}(k) - f_{2n+2}(k), 2x)_X \end{aligned}$$

Since $M(f_{2n} - f_{2n+2}, 2x) \geq \min\{M(f_{2n} - f_{2n+1}, x), M(f_{2n+1} - f_{2n+2}, x)\}$,

and $M(f_{2n} - f_{2n+1}, x)_Y \geq \inf_{k \in I} M(f_{2n}(k) - f_{2n+1}(k), x)_X$, we get

$$M(f_{2n+1} - f_{2n+2}, qx)_Y \geq M(f_{2n} - f_{2n+1}, x)_Y$$

Similarly,

$$M(f_{2n+2} - f_{2n+3}, qx)_Y \geq M(f_{2n+1} - f_{2n+2}, x)_Y$$

Thus, in general

(4) $$M(f_n - f_{n+1}, qx)_Y \geq M(f_{n-1} - f_n, x)_Y,\ n = 1, 2, 3, \ldots$$

Therefore $\{f_n\}$ is a Cauchy sequence (By lemma 1.1) converging to some f* in Y.

Let $U_{Pf^*}(\in, r)$ be a neighbourhood of Pf*. Since $f_n \to f^*$, for $\in, r > 0$, there exists $N(\in, r)$ such that $n \geq N$ implies

(5) $M(f^* - f_{2n+2}, (1 - q)\in/2q)_Y > 1 - \lambda$

and $M(f_{2n+1} - f_{2n+2}, (1 - q)\in/2q)_Y > 1 - \lambda$

Now by (1)

$$\begin{aligned} M(Pf^* - f_{2n+2}(k), \in)_X &= M(Pf^* - Qf_{2n+1}, \in)_X \\ &\geq \min\{M(f^* - f_{2n+1}, \in/q)_Y, M(f^*(k) - Pf^*, \in/q)_X, M(f_{2n+1}(k) - Qf_{2n+1}, \in/q)_X, \\ &\quad M(f_{2n+1}(k) - Pf^*, 2\in/q)_X, M(f^*(k) - Qf_{2n+1}, 2\in/q)_X\} \\ &\geq \min\{M(f^* - f_{2n+1}, \in/q)_Y, M(f^*(k) - f_{2n+2}(k), (1 - q)\in/2q), \\ &\quad M(f_{2n+2}(k) - Pf^*, (1 - q)\in/2q)_Y, M(f_{2n+1}(k) - f_{2n+2}(k), \in/q)_Y, \\ &\quad M(f_{2n+1}(k) - f_{2n+2}(k), \in/q)_Y, M(f_{2n+2}(k) - Pf^*, \in/q)_Y, \end{aligned}$$

$$M(f^*(k) - f_{2n+2}(k), 2\in/q)_Y\} \geq 1 - \lambda, \text{ by (5)}$$

Hence $\{f_n\} \to Pf^*$ implying $Pf^* = f^*(k)$. Similarly $Qf^* = f^*(k)$. Thus $Pf^* = f^*(k) = Qf^*$.

To prove (B), we have $f_n - g_n \in S$ for $n = 0, 1, 2, \ldots$

$$M(f_n - g_n, \in)_Y \geq \min \{f_n - f_{n+1}, \in/3)_Y, M(f_{n+1} - g_{n+1}, \in/3)_Y, M(g_{n+1} - g_n, \in/3)_Y\}$$

Repeated use of (4) gives

$$M(f_n - g_n, \in)_Y \geq \min \{M(f_0 - f_1, \in/3\, q^n)_Y, M(f_1 - g_1, \in/3\, q^n)_Y, M(g_1 - g_0, \in/3q^n)_Y\}.$$

Since each term on the R.H.S. $\to 1$ as $n \to \infty$. Therefore, $\lim f_n = \lim g_n$. This proves (B). If f^* and $g^* \in S_f^*$ and $f^* \neq g^*$, then by (1),

$$\text{-} \quad M(f^* - g^*, \in)_Y = M(f^*(k) - g^*(k), \in)_X = M(Pf^* - Qg^*, \in)_Y \geq M(f^* - g^*, \in/q)_Y, \text{ a contradiction.}$$

Thus P and Q have a unique common fixed point in S_f^*.

REFERENCES

Bocsan Gh (1974). On some fixed point theorem in random normed space, *Univ. din Timisoara, Seminarul de teoria functillor si matematici aplicate, a Spatii metrice probabiliste* **18**.

Chamola KP (1988). Fixed points of mappings satisfying a new contraction condition in Random Normed spaces, *Mathematica Japonica* **33(**6) 821-825.

Chugh R and Rathi S (2005). Weakly Compatible maps in Fuzzy Metric Space, *Universitatea Din Bacau, Studii Si Cercetari Stiintifice Seria: Matematica* **15** 31-38.

Erceg MA (1979). Metric space in fuzzy set theory. *Journal of Mathematical Analysis and Applications* **69** 205-230.

George MA and Veeramani P (1994). Some results in fuzzy metric space. *Fuzzy Sets and Systems* **64** 395-399.

Grabiec M (1983). Fixed point in fuzzy metric spaces. *Fuzzy Sets and System* **27** 385-389.

Kaleva O and Seikkala O (1984). On fuzzy metric spaces. *Fuzzy Sets and System* **12** 215.

Kramosil J and Michalek J (1975). Fuzzy metric and statistical metric spaces. *Kybernetica* **11** 330-334.

Menger K (1942). Statistical Metrics. *Proceedings of the National Academy of Sciences USA* **28** 535-537.

Schweizer B and Sklar A (1983). Probabilistic metric spaces. *North Holland Series* (1983).

Serstnev AN (1963). The notion of random normed spaces, *Doklady Akademii Nauk USSR* **149** 280-283.

Subrahmanyam PV (1995). Common fixed point theorem in fuzzy metric spaces. *Information Science* **83**(3-4) 109-112.

Zadeh L (1965). Fuzzy Sets. *Information Control* **89** 338-353.