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Research Article

PROPER SOLUTION TO GENERATE SEQUENCES FOR SET OF PRIMITIVE PYTHAGOREAN TRIPLES

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ABSTRACT
We know already that the set of positive integers, which are satisfying the Pythagoras equation of three variables are called Pythagorean triples. The all unknowns in this Pythagorean equation have already been solved by mathematicians Euclid and Diophantine. However, the solution defined by Euclid & Diophantine is also again having unknowns. The only possible to solve the Pythagorean equations is trial & error method and it will not be so practical and easy especially for time bound works, since the Pythagorean equations are having more than two unknown variables. The scope of work is to generate sequences of the primitive Pythagorean triples without missing any set of the triples, by single simple formula. After conducting various iterations, the author has developed a simple formula to generate plenty of sequences and plenty of terms in each sequence for set of primitive Pythagorean triples. The formula has been proved with an appropriate example. It is very useful for Students, Research scholars, Engineers and persons whose work is related to right-angled triangles, rectangular prisms and Number theory.

***Key Words**: Right Triangle, Pythagoras Theorem, Number Theory and Primitive Pythagorean Triples.*

INTRODUTION
The *Right angled triangles* (Weisstein Eric, 2003) are having significant role in the study of geometry. In any right triangle, one of the interior *vertex angles* (Weisstein Eric, 2003) is 90°. The longest of all three sides of the right triangle is called *hypotenuse* (Weisstein Eric, 2003). The right triangle has been used in trades for thousands of years. Ancient Egyptians found that they could always get a square corner using the 3-4-5 right triangle. Carpenters still use the 3-4-5 triangle to square corners. Later, a Greek mathematician named Pythagoras who lived about 2500 years ago developed the most famous formula to find the side lengths of any right triangles, possibly uses in all of mathematics. He proved that, for a right triangle, the sum of the squares of the two sides that join at a right angle equals the square of the third side, which is the side opposite the right angle is called the hypotenuse of the right triangle. The two shorter sides are usually called legs. The set of positive integers, which are satisfying the Pythagoras equation of three variables are called Pythagorean triples. The Pythagoras theorem has many uses in astronomical survey, trigonometry, application in Engineering and etc. Surveyors use it in their work. In any engineering work, the use of right triangle is very important. Using the *trigonometry* (Weisstein Eric W, 2003, p.3052), the airport cloud ceiling can be determined by the airport meteorologists to ensure the safety of the flights. The *primitive Pythagorean triples* (Weisstein Eric, No date) are set of three numbers which are satisfying the Pythagorean equation as well as there is no common factors among them other than unity.

EXISTING FORMULAE AND METHODS
Pythagorean triples are a set of integers (x, y, r), which are satisfying the Pythagoras equation $x^2 + y^2 = r^2$. This equation is specifically applicable for right angled triangle (Fig.1). The unknowns in the Pythagorean equation have already been solved by mathematicians *Euclid and Diophantine*
(Gellert *et al.*, 1989). The solution defined by mathematicians Euclid & Diophantine for Pythagorean triples as “All Pythagorean triples in which ‘x’, ‘y’, ‘r’ are without common factor and ‘x’ is odd are obtained by replacing the letters ‘a’ and ‘b’ in the triple $\{(a^2 - b^2), (2ab), (a^2 + b^2)\}$ by whole numbers

that have odd sum and no common factor". However the solution defined by Euclid & Diophantine is also again having unknowns. However, the solution defined by Euclid & Diophantine is also again having unknowns. The existing methods are not much suitable to generate the complete set of primitive Pythagorean triples. The only possible to solve the Pythagorean equations was trial & error method. Moreover, the trial & error method to obtain these values are not so practical and easy especially for time bound works, since the Pythagorean equations are having more than two unknown variables. The method to generate Primitive Pythagorean triples was already defined by Fibonacci and Dickson.

ANALYSIS AND FORMULATION FOR FORMULA

The primitive Pythagorean triples are set of three numbers which are satisfying the Pythagorean equation as well as there is no Greatest Common Divisor (GCD) (Weisstein Eric W, 2003, p.1257) among them other than unity. The values of integer numbers for x, y, r is a set of Pythagorean triples for Pythagorean equation $x^2 + y^2 = r^2$. The equation satisfies the right-angled triangle (see fig. 1).

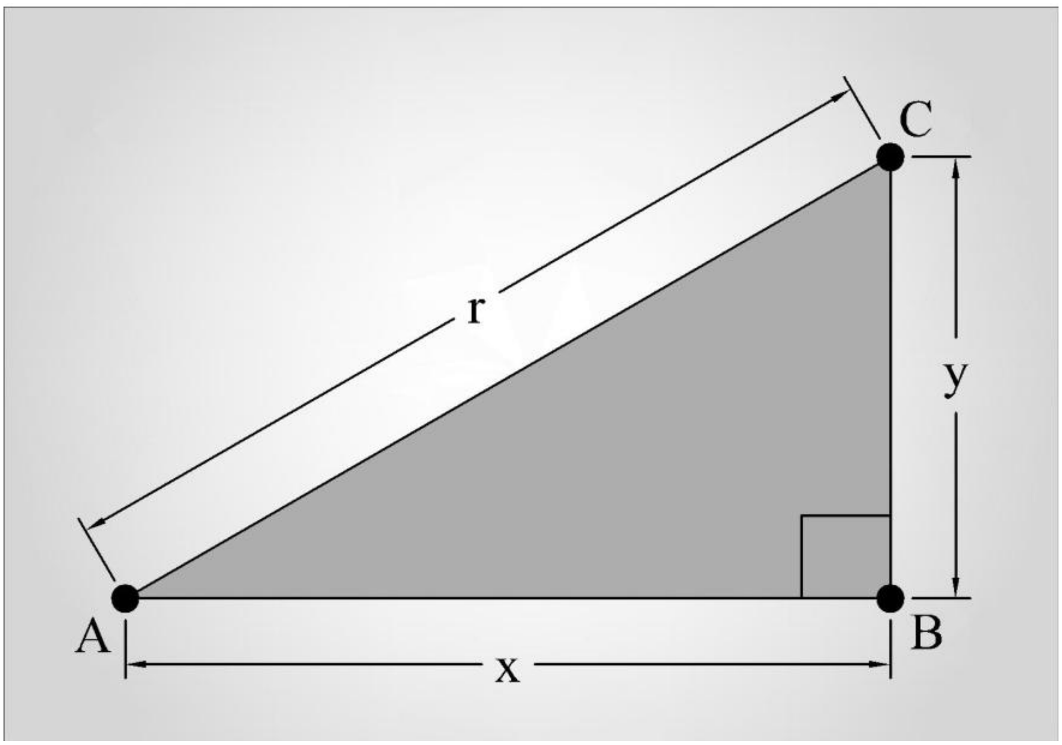

Fig.1: A diagram of Right angled triangle ABC

Let (x, y, r) be a set of Pythagorean triples for Pythagorean equation $x^2 + y^2 = r^2$. After conducting several iterations by the author, the triples are sorted out depends upon its certain common properties such as $r - y$ and $r - x$ are particular constant for each group. The set of primitive Pythagorean triples, which are available in online *"http://www.tsm-resources.com/alists/trip.html"* (Douglas Butler, No date) are taken into account for this analysis. The mathematical relation between these value are studied carefully and these are sorted out into several groups depending upon the value of $r - y$ and $r - x$.

Let, $p_m = r - y = (2m - 1)^2$ and $q_n = r - x = 2n^2$. Where, 'm' and 'n' are any natural integer numbers. Finally these groups have been formulated to a single formula.

The classified set of values of primitive Pythagorean triples (x, y, r) are tabulated as under:

m	p_m	n =	1	2	3	4	5	6	7	8	9	10	n
		$q_n =$	2	8	18	32	50	72	98	128	162	200	$2n^2$
1	1	x =	3	5	7	9	11	13	15	17	19	21	*
		y =	4	12	24	40	60	84	112	144	180	220	*
		$r =$	5	13	25	41	61	85	113	145	181	221	*
2	9	x =	15	21	-	33	39	-	51	57	-	69	*
		y =	8	20	-	56	80	-	140	176	-	260	*
		$r =$	17	29	-	65	89	-	149	185	-	269	*
3	25	x =	35	45	55	65	-	85	95	105	115	-	*
		y =	12	28	48	72	-	132	168	208	252	-	*
		$r =$	37	53	73	97	-	157	193	233	277	-	*
4	49	x =	63	77	91	105	119	133	-	161	175	189	*
		y =	16	36	60	88	120	156	-	240	288	340	*
		$r =$	65	85	109	137	169	205	-	289	337	389	*
5	81	x =	99	117	-	153	171	-	207	225	-	261	*
		y =	20	44	-	104	140	-	224	272	-	380	*
		$r =$	101	125	-	185	221	-	305	353	-	461	*
6	121	x =	143	165	187	209	231	253	275	297	319	341	*
		y =	24	52	84	120	160	204	252	304	360	420	*
		$r =$	145	173	205	241	281	325	373	425	481	541	*
7	169	x =	195	221	247	273	299	325	351	377	403	429	*
		y =	28	60	96	136	180	228	280	336	396	460	*
		$r =$	197	229	265	305	349	397	449	505	565	629	*
8	225	x =	255	285	-	345	-	-	435	465	-	-	*
		y =	32	68	-	152	-	-	308	368	-	-	*
		$r =$	257	293	-	377	-	-	533	593	-	-	*
9	289	x =	323	357	391	425	459	493	527	561	595	629	*
		y =	36	76	120	168	220	276	336	400	468	540	*
		$r =$	325	365	409	457	509	565	625	689	757	829	*
10	361	x =	399	437	475	513	551	589	627	665	703	741	*
		y =	40	84	132	184	240	300	364	432	504	580	*
		$r =$	401	445	493	545	601	661	725	793	865	941	*
11	441	x =	483	525	-	609	651	-	-	735	777	-	*
		y =	44	92	-	200	260	-	-	392	464	-	*
		$r =$	485	533	-	641	701	-	-	833	905	-	*
12	529	x =	575	621	667	713	759	805	851	897	943	989	*
		y =	48	100	156	216	280	348	420	496	576	660	*
		$r =$	577	629	685	745	809	877	949	1025	1105	1189	*
m	$(2m-1)^2$	$x = p_m + \sqrt{2p_m q_n}$											*
		$y = q_n + \sqrt{2p_m q_n}$											*
		$r = p_m + q_n + \sqrt{2p_m q_n}$											*

Table 1: Values of set of Pythagorean triples (x, y, r)

The above set of values refers that

$3^2+4^2=5^2, 5^2+12^2=13^2, 7^2+24^2=25^2, \dots, x^2+y^2=r^2$
$15^2+8^2=17^2, 21^2+20^2=29^2, 33^2+56^2=65^2, \dots, x^2+y^2=r^2$
$35^2+12^2=37^2, 45^2+28^2=53^2, 55^2+48^2=73^2, \dots, x^2+y^2=r^2$ and etc.

Finally a formula which will be exactly appropriate to generate set of primitive Pythagorean triples at mth row and nth column for the Pythagorean equation $x^2+y^2=r^2$ is formulated in matrix form as follows

$$\begin{pmatrix} x_{mn} \\ y_{mn} \\ r_{mn} \end{pmatrix} = \begin{pmatrix} p_m + \sqrt{2p_m q_n} \\ q_n + \sqrt{2p_m q_n} \\ p_m + q_n + \sqrt{2p_m q_n} \end{pmatrix} \quad \text{-------------- [1]}$$

Where, $p_m=(2m-1)^2$, $q_n=2n^2, m=1,2,3,4,\dots,\infty$ and $n=1,2,3,4,\dots,\infty$. However, if there is a GCD other than unity for p and q of any particular set, it is not primitive triples.

Substituting values of $1 \le m \le \infty$ various sequences can be developed. For each sequences substituting $1 \le q \le \infty$, the various elements of the sequence can be developed continuously.

m or n	$p=(2m-1)^2$	$q=2n^2$
1	1	2
2	9	8
3	25	18
4	49	32
5	81	50
6	121	72
7	169	98
8	225	128
9	289	162
10	361	200
11	441	242
12	529	288
13	625	338
14	729	392
15	841	450
16	961	512
17	1089	578
18	1225	648
19	1369	722
20	1521	800
…	…	…

Table 2: Values of for p and q for Pythagorean triples

These are the necessary procedure to generate any sequences of primitive Pythagorean triples. By using the formula (eqn.1), we can calculate plenty of set of primitive Pythagorean triples in each sequence.
The GCD greater than unity are identified for specific values for ready reference as tabulated in table 3 for ready reference.

Note: The symbol ○ refers that there is GCD other than unity for those two numbers and the symbol ● refers that there is no GCD other than unity for those two numbers.

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RESULT AND DISCUSSION

EXAMPLE-1

The formulae for primitive Pythagorean triples (x_n, y_n, r_n) for $m = 21$ and for $n = 1$ to 10 can be calculated by the formula:

	2	8	18	32	50	72	98	128	162	200	242	288	338	392	450	512	578	648	722	800	882	968	1058	1152	1250
1	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
9	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●
25	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○
49	●	●	●	●	●	●	○	●	●	●	●	●	●	○	●	●	●	●	●	●	○	●	●	●	●
81	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●
121	●	●	●	●	●	●	●	●	●	●	○	●	●	●	●	●	●	●	●	●	●	○	●	●	●
169	●	●	●	●	●	●	●	●	●	●	●	●	○	●	●	●	●	●	●	●	●	●	●	●	●
225	●	●	○	●	○	○	●	●	○	○	●	○	●	●	○	●	●	○	●	○	○	●	●	○	○
289	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	○	●	●	●	●	●	●	●	●
361	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	○	●	●	●	●	●	●
441	●	●	○	●	●	○	○	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●
529	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	○	●	●
625	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○	●	●	●	●	○
729	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●	●	○	●
841	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
961	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
1089	●	●	○	●	●	○	●	●	○	●	○	○	●	●	○	●	●	○	●	●	○	○	●	○	●
1225	●	●	●	●	○	●	○	●	●	○	●	●	●	○	○	●	●	●	●	○	○	●	●	●	○
1369	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
1521	●	●	○	●	●	○	●	●	○	●	●	○	○	●	○	●	●	○	●	●	○	●	●	○	●
1681	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
1849	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
2025	●	●	○	●	○	○	●	●	○	○	●	○	●	●	○	●	●	○	●	○	○	●	●	○	○
2209	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●

Table 3: The GCD for p upto 1250 and q upto 2209

$$\begin{pmatrix} x \\ y \\ r \end{pmatrix} = \begin{pmatrix} p_m + \sqrt{2p_m q_n} \\ q_n + \sqrt{2p_m q_n} \\ p_m + q_n + \sqrt{2p_m q_n} \end{pmatrix} \quad \text{-------------------------} [2]$$

Substituting $p_{21} = 1681$ in above formula and $q = 2, 8, 18, \ldots$ in above eqn., we get

$$\begin{pmatrix} x \\ y \\ r \end{pmatrix} = \begin{pmatrix} 1681 + \sqrt{2 \times 1681 \times q_n} \\ q_n + \sqrt{2 \times 1681 \times q_n} \\ 1681 + q_n + \sqrt{2 \times 1681 \times q_n} \end{pmatrix} \quad \text{-------------------------} [3]$$

Substituting n = 1 to 10, we get the values as in the form of matrix.

$$\begin{pmatrix} x \\ y \\ r \end{pmatrix} = \begin{pmatrix} 1763 & 1845 & 1927 & 2009 & 2091 & 2173 & 2255 & 2337 & 2419 & 2541 & \cdots \\ 84 & 172 & 264 & 360 & 460 & 564 & 672 & 784 & 900 & 1020 & \cdots \\ 1765 & 1853 & 1945 & 2041 & 2141 & 2245 & 2353 & 2465 & 2581 & 2701 & \cdots \end{pmatrix}$$

Similarly, the formula for the other sequence can be generated.

CONCLUSION

In this article the existing formulae and method have been shown for the reference in order to understand the practical problem when solving the Pythagorean equation with primitive Pythagorean triples. A new formula to generate a sequence of primitive Pythagorean triples for which has been developed now have been defined and proved with two appropriate examples for detailed explanation. The eqn. 1 is the necessary formula for generating any sequences for set of Pythagorean triples. By the formula, we can generate plenty of sequences and each sequence is of infinitive terms of set of Pythagorean triples. A table of GCD for various values of p upto 2209 and q upto 1250 has also been provided. The formulae, which have been defined in this article will be one of the proper solutions to generate a sequence for set of primitive Pythagorean triples and also useful for those doing research or further study in the *Diophantine equations* (Eric Weisstein, 2003) and *Number theory* (Eric Weisstein, 2003) especially this formula will be as a clear road map to solve the problem of *Perfect Cuboid* (Eric Weisstein, 2012) which is having dimensions of all sides and diagonals with integers.

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