International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2012 Vol. 2 (4) October-December, pp.27-40/Aghili and Zeinali
Research Article

INTEGRAL TRANSFORM METHODS FOR SOLVING FRACTIONAL PDES AND EVALUATION OF CERTAIN INTEGRALS AND SERIES

***A. Aghili and H. Zeinali**
**Department of Applied Mathematics, University of Guilan Rasht-Iran*
**Author for Correspondence*

ABSTRACT
In this work, the authors implemented two dimensional Laplace transform to evaluate certain integrals, series and solving partial fractional differential equations. Constructive examples are also provided to illustrate the ideas. The result reveals that the integral transform method is very effective and convenient.
Mathematics Subject Classification 2010: 26A33; 34A08; 34K37; 35R11.

Key Words: *Non-Homogeneous Time Fractional Heat Equations, Laplace Transform, Kelvin's Function and Time Fractional Wave Equations*

INTRODUCTION
The object of present work is to extend the application of Laplace transform to derive analytical solution for certain boundary value problem of wave equation. In 1985 Dahiya considered a method of computing Laplace transform pairs of n-dimensions and prove a theorem to obtain new n-dimensional Laplace transform pairs. Later in 1990 Dahiya and Vinayagamoorthy established several new theorems and corollaries for calculating Laplace transform pairs of n-dimensions. They also considered two boundary value problems. The first was related to Heat transfer for cooling off a very thin semi-infinite homogenous plate into the surrounding medium solved by using double Laplace transform .The second, was Heat equation for the semi-infinite slab where the sides of the slab are maintained at prescribed temperature.
In 1992 Saberi Najafi and Dahiya established several new theorems for calculating Laplace theorems of n-dimensions and in the second part application of those theorems to a number of commonly used special functions was considered, and finally, one-dimensional wave equation involving special functions was solved by using two dimensional Laplace transform.
Later in 1999 Dahiya proved certain theorems involving the classical Laplace transform of N-variables and in the second part a non-homogenous partial differential equations of parabolic type with some special source function was considered.
Recently in 2004, 2006, 2008 the authors established new Theorems and corollaries involving systems of two-dimensional Laplace transforms containing several equations.
The generalization of the well-known Laplace transform

$$L\{f(t);s\}=\int_0^\infty e^{-st}f(t)dt$$

to n-dimensional is given by

$$L_n\{f(\bar{t});\bar{s}\}=\int_0^\infty\int_0^\infty\ldots\int_0^\infty \exp\left(-\bar{s}\,\bar{t}\right)f(\bar{t})P_n\left(d\bar{t}\right),$$

where $\bar{t}=\left(t_1,t_2,\ldots,t_n\right)$, $\bar{s}=\left(s_1,s_2,\ldots,s_n\right)$, $\bar{s}\,\bar{t}=\sum_{i=1}^{n}s_it_i$, and $P_n\left(d\bar{t}\right)=\prod_{k=1}^{n}dt_k$.

In this paper we focus on two dimensional Laplace transform of the function $f(x,y)$, which is defined as

$$F(p,q)=\int_0^{+\infty}\int_0^{+\infty}e^{-px-qy}f(x,y)dxdy.$$

Definition.1.1: The inverse of two dimensional Laplace transforms $F(p,q)$ is defined by

$$f(x,y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}e^{qy}F(p,q)dq\right)dp,$$

in which $\operatorname{Re}p<c,\operatorname{Re}q<c'$.

Inversion of Two Dimensional Laplace Transforms

In this section, an algorithm to invert one and two dimensional Laplace transforms
is presented.

Theorem.2.1 (Efros Theorem):

Let $L\{f(t)\}=F(s)$ and $L\{u(t,\tau)\}=U(s)\exp(-\tau q(s))$ and assume $U(s),q(s)$ are analytic, then by taking Laplace transform w.r.t t, one has

$$L\{\int_0^{+\infty}f(\tau)u(t,\tau)d\tau;t\to s\}=U(s)F(q(s)).$$

Proof: See (Ditkin 1962).

Lemma.2.2 (Schouten-Van der Pol): Consider a function f(t) which has the Laplace transform F(s) which is analytic in the half plane Re(s)>s0. If q(s) is also analytic for Re(s)>s0, then the inverse of F (q(s)) is as following

$$L^{-1}\{F(q(s));s\to t\}=\int_0^{\infty}f(\tau)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-q(s)\tau}e^{is}ds\right]d\tau.$$

Special case: $q(s)=\sqrt{s}$;

$$L^{-1}\{F(\sqrt{s});s\to t\}=\frac{1}{2t\sqrt{\pi t}}\int_0^{\infty}\tau f(\tau)\exp(-\frac{\tau^3}{4t})d\tau.$$

Proof: See (Duffy 2004).

In this section, a new class of inverse Laplace transforms of exponential functions involving nested square roots is determined. Inverse Laplace transforms involving nested square roots arise in many areas of applied Mathematics, usually as a result of linear evolution partial differential equations of forth order in the spatial variables. Examples of such problems abound in fluid mechanics.

Lemma 2.3: The following relation holds true.

$$L^{-1}\{\exp(-x\sqrt{2s+2\sqrt{s^2-1}})\}=\frac{x}{2\sqrt{\pi}}\int_0^t\eta(t-\eta)^{-\frac{3}{2}}\exp(\eta-2t-\frac{x^2t}{4\eta(t-\mu)})d\eta$$

Proof: Let us assume that $F(s)=\exp(-x\sqrt{2s+2\sqrt{s^2-1}})$ and then the exponent can be re-written as follows,

$$\exp(-x\sqrt{2s+2\sqrt{s^2-1}})=\exp(-x\sqrt{s+1}).\exp(-x\sqrt{s-1}).$$

On the other hand, from Laplace table we have

$$L^{-1}\{\exp(-x\sqrt{s+a})\}=\frac{xt^{-3/2}e^{-(at+\frac{x^2}{4t})}}{2\sqrt{\pi}},$$

So that,

$$L^{-1}\{F(s)\} = L^{-1}\{\exp(-x\sqrt{2s+2\sqrt{s^2-1}})\} =$$
$$= L^{-1}\{\exp(-x\sqrt{s+1})\} * L^{-1}\{\exp(-x\sqrt{s-1})\}$$
$$= \frac{x}{2\sqrt{\pi}}\int_0^t \eta(t-\eta)^{-\frac{3}{2}} \exp(\eta - 2t - \frac{x^2 t}{4\eta(t-\mu)})d\eta.$$

Example 2.4: Suppose that $F(p,q)$ be the two dimensional Laplace transform of the function $f(x,y)$

$$F(p,q) = \frac{1}{(q+\kappa\sqrt{\lambda+\sqrt{p}})(\sqrt{p}+\mu)}, \quad \lambda,\mu \in R, \tag{2.1}$$

Then we have

$$f(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{\mu+\sqrt{p}}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{qy}}{q-(-\kappa\sqrt{\lambda+\sqrt{p}})}dq\right)dp =$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{\mu+\sqrt{p}} e^{-y\kappa\sqrt{\lambda+\sqrt{p}}}dp. \tag{2.2}$$

Let us define the function

$$G(p,q) = \frac{e^{-y\kappa\sqrt{\lambda+p}}}{p+\mu},$$

It means that

$$F(p,q) = G(\sqrt{p},q),$$

It suffices to find the inverse of $G(p,q)$ and then using Efros theorem to obtain the inverse of $F(p,q)$.
One may use the following well known integral representation

$$\int_0^\infty e^{-a^2u^2-\frac{b^2}{u^2}}du = \frac{\sqrt{\pi}}{2a}e^{-2ab}. \tag{2.3}$$

Making a change of variable to get

$$-y\kappa\sqrt{p+\lambda} = -2ab \to a=1, b=\frac{y\kappa\sqrt{p+\lambda}}{2},$$
$$e^{-y\kappa\sqrt{p+\lambda}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{y^2\kappa^2(p+\lambda)}{4u^2}}du,$$

Now, we can evaluate the inverse of $G(p,q)$ as bellow

$$g(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{p+\mu}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{y^2\kappa^2(p+\lambda)}{4u^2}}du\right)dp$$
$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{p+\mu}e^{-\frac{y^2\kappa^2(p+\lambda)}{4u^2}}dp\right)du,$$

by change of variable $p+\mu=s$,

$$g(x,y)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{y^2\kappa^2\lambda}{4u^2}}\left(\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty}\frac{e^{(s-\mu)x}}{s}e^{-\frac{y^2\kappa^2(s-\mu)}{4u^2}}ds\right)du$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\mu x-u^2-\frac{y^2\kappa^2(\lambda-\mu)}{4u^2}}\left(\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty}\frac{e^{-(\frac{y^2\kappa^2}{4u^2})s}e^{sx}}{s}ds\right)du=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\mu x-u^2-\frac{y^2\kappa^2(\lambda-\mu)}{4u^2}}\delta(x-(\frac{y^2\kappa^2}{4u^2}))du.$$

By introducing a new change of variable, $x-(\frac{y^2\kappa^2}{4u^2})=w$,we obtain

$$g(x,y)=\frac{e^{-\mu x}}{2\sqrt{\pi}}\int_0^\infty e^{-(w+x)(\lambda-\mu)-\frac{y^2\kappa^2}{4(x+w)}}(y\kappa)(x+w)^{\frac{-3}{2}}\delta(w)dw=\frac{y\kappa e^{-\frac{y^2\kappa^2+4x^2\lambda}{4x}}}{2x\sqrt{\pi x}}.$$

At this point, using Efros theorem for $q(s)=\sqrt{s}$ and lemma 2.2 leads to the following

$$f(x,y)=L^{-1}\{G(\sqrt{p},q)\}$$

$$=\int_0^{+\infty}\left(\frac{te^{-\frac{t^2}{4x}}}{\sqrt{4\pi x^3}}\right)\left(\frac{y\kappa e^{-\frac{y^2\kappa^2+4t^2\lambda}{4t}}}{2t\sqrt{\pi t}}\right)dt=\frac{y\kappa}{8\pi x\sqrt{x}}\int_0^{+\infty}\frac{e^{(\frac{-t^2}{4x}+\frac{y^2\kappa^2+4t^2\lambda}{4t})}}{\sqrt{t}}dt.$$

Applications of Two Dimensional Laplace Transform

The multi dimensional Laplace transform is used frequently in engineering and physics. The two dimensional Laplace transform can also be used to solve partial differential equations and is used extensively in electrical engineering. The two dimensional Laplace transforms reduces a linear partial differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The PDE can then be solved by applying the inverse two dimensional Laplace transform.

There are also some applications of two dimensional Laplace transform to evaluate certain integrals and series as is discussed in the following paragraphs.

3.1. Evaluation of certain series:

Suppose that $F(p,q)$ be the two dimensional Laplace transform of the function $f(x,y)$ as below

$$F(p,q)=\int_0^{+\infty}\int_0^{+\infty}e^{-px-qy}f(x,y)dxdy,$$

Taking $p=q$ leads us to the following relationship

$$F(p,p)=\int_0^{+\infty}\int_0^{+\infty}e^{-p(x+y)}f(x,y)dxdy.$$

By making a change of variable, $x+y=w$ one has

$$F(p,p)=\int_0^{+\infty}\int_y^{+\infty}e^{-pw}f(w-y,y)dwdy=\int_0^{+\infty}e^{-pw}\int_0^w f(w-y,y)dydw.$$

Now letting $p = n$, to get

$$F(n,n) = \int_0^{+\infty} e^{-nw}\left(\int_0^w f(w-y,y)dy\right)dw,$$

Summing over n to obtain

$$\sum_{n=1}^{\infty} F(n,n) = \int_0^{\infty}\left(\sum_{n=1}^{\infty} e^{-nw}\right)\left(\int_0^w f(w-y,y)dy\right)dw.$$

The above relation can be re- written as

$$\sum_{n=1}^{\infty} F(n,n) = \int_0^{+\infty} \frac{g(w)}{e^w - 1}dw,$$

in which

$$g(w) = \int_0^w f(w-y,y)dy.$$

Example 3.1.1: Evaluate the following series

$$S = \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2}.$$

Solution by using the above procedure, we get

$$F(p,q) = \frac{4}{(2p-1)(2q-1)} = \frac{1}{p-\frac{1}{2}} \times \frac{1}{q-\frac{1}{2}} = L_2\{e^{\frac{x+y}{2}}\}.$$

Then we have

$$g(w) = \int_0^w e^{w/2}dy = we^{w/2},$$

consequently

$$\sum_{n=1}^{\infty} F(n,n) = \int_0^{\infty} \frac{we^{w/2}}{e^w - 1}dw = \frac{1}{2}\int_0^{+\infty} \frac{w}{\sinh\frac{w}{2}}dw.$$

On the other hand, we know that (or by calculus of residues)

$$\int_0^{+\infty} \frac{x}{\sinh ax}dx = \frac{\pi^2}{4a^2}.$$

Therefore the final solution is

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)^2} = \frac{\pi^2}{2}.$$

3.2 Evaluation of the integrals:

In applied mathematics, the Kelvin functions Berν(x) and Beiν(x) are the real and imaginary parts, respectively, of

$$J_\nu(xe^{3\pi i/4}),$$

Where x is real, and Jν(z), is the νth order Bessel function of the first kind. Similarly, the functions Kerν(x) and Keiν(x) are the real and imaginary parts, respectively, of $K_\nu(xe^{\pi i/4})$, where Kν(z), is the νth order modified Bessel function of the second kind. These functions are named after William

Thomson,1st Baron Kelvin. The Kelvin functions were investigated because they are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics.

One of the main applications of two dimensional Laplace transform is evaluating the integrals as discussed in the following.

Lemma 3.2.1: Show the following integral relations

$$1-\int_0^{+\infty} ber(2\sqrt{\lambda})\cos a\lambda d\lambda = -\frac{\pi}{2a}e^{\frac{1}{a}}.$$

$$2-\int_0^{+\infty} \frac{1}{\lambda} bei(2\sqrt{\lambda})\sin\theta\lambda d\lambda = \frac{\pi}{2}(\theta-1).$$

Proof1: Let us define the following function

$$I(a,x,y) = \int_0^{+\infty} ber(2\sqrt{xy\lambda})\cos a\lambda \ d\lambda,$$

By using the formula

$$L_2\{ber(2\sqrt{axy});x \to p, y \to q\} = \frac{pq}{p^2q^2+a^2},$$

We have

$$L_2\{I(a,x,y)\} = \int_0^{+\infty} \frac{pq}{\lambda^2+p^2q^2}\cos a\lambda d\lambda.$$

Now, with the aid of Fourier transform we get the following relationship

$$I(a,x,y) = \frac{\pi}{2}e^{-apq}.$$

However, it leads to

$$I(a,x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\pi}{2}e^{-(ap)q}e^{qy}dq\right)dp = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{2}e^{px}\delta(y-ap)dp.$$

Now, let us make a change of variable $y-ap=w$ and consequently $-adp=dw$

$$I(a,x,y) = -\frac{\pi}{2a}\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty} e^{(\frac{y-w}{a})x}\delta(w)dw = -\frac{\pi}{2a}e^{\frac{xy}{a}}.$$

Letting $x=y=1$one gets

$$\int_0^{+\infty} ber(2\sqrt{\lambda})\cos a\lambda d\lambda = -\frac{\pi}{2a}e^{\frac{1}{a}}.$$

2 – Let us consider the function $I(\theta,x,y) = \int_0^{+\infty}\frac{1}{\lambda}bei(2\sqrt{\lambda xy})\sin\theta\lambda d\lambda.$

By taking two dimensional Laplace transform of the above relation with respect to x, y we get

$$L_2\{I(\theta,x,y)\} = \bar{I}(\theta) = \int_0^{+\infty}\frac{\sin\theta\lambda}{\lambda}\frac{1}{\lambda^2+p^2q^2}d\lambda,$$

on the other hand, we know that

$$\bar{I}(0)=0,\bar{I}'(\theta)=\int_0^{+\infty}\frac{\cos\theta\lambda}{\lambda^2+(pq)^2}d\lambda=\frac{\pi}{2pq}e^{-\theta pq},$$

consequently

$$\bar{I}(\theta)=-\frac{\pi}{2p^2q^2}e^{-\theta pq},$$

which leads to

$$I(\theta,x,y)=-\frac{\pi}{2}\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p^2}e^{px}\Big(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{-\theta pq}}{q^2}e^{qy}dq\Big)dp\Big)$$
$$=-\frac{\pi}{2}\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p^2}e^{px}(y-\theta p)dp\Big)=-\frac{\pi}{2}(xy-\theta H(x)).$$

By setting $x=y=1$, one gets

$$\int_0^{+\infty}\frac{1}{\lambda}bei(2\sqrt{\lambda})\sin\theta\lambda d\lambda=\frac{\pi}{2}(\theta-1).$$

Lemma 3.2.2: The following relationship for $a\in R_+, m,n\in N$ holds true

$$L_2^{-1}\Big\{\frac{(pq)^{n-m}}{(pq)^n-a^n}\Big\}=\frac{1}{na^{m-1}}\sum_{k=0}^{n-1}\frac{I_0(2e^{k/2}\sqrt{axy})}{e^{(m-1)k}}.$$

Proof: see [Ditkin 1979].

Lemma 3.2.3: Evaluate the following integral

$$\int_0^{+\infty}\sin\sqrt{t}\,\sin\frac{1}{\sqrt{t}}\frac{dt}{t}.$$

Proof: Let us define the following function

$$f(x,y)=\int_0^{+\infty}\sin x\sqrt{t}\,\sin\frac{y}{\sqrt{t}}\frac{dt}{t},$$

Now by taking two dimensional Laplace transform we get

$$F(p,q)=\int_0^{+\infty}\frac{dt}{t}\left[\frac{\sqrt{t}}{p^2+t}.\frac{1/\sqrt{t}}{q^2+\frac{1}{t}}\right]=\frac{1}{q^2}\int_0^{+\infty}\frac{dt}{(t+p^2)(t+\frac{1}{q^2})},$$

and consequently

$$\frac{1}{p^2q^2-1}\int_0^{+\infty}\Big(\frac{1}{t+\frac{1}{q^2}}-\frac{1}{t+p^2}\Big)dt=\frac{1}{p^2q^2-1}\ln\frac{t+\frac{1}{q^2}}{t+p^2}\Bigg|_0^{+\infty}.$$

Finally one gets

$$F(p,q)=(\frac{2pq}{p^2q^2-1})(\frac{1}{pq}\ln pq).$$

Now, it remains to use inverse of Laplace transform to get $f(x,y)$. To do that, we use lemma 3.2.2 and convolution theorem as bellow

$$\alpha(x,y)=L^{-1}\{\frac{1}{pq}\ln pq\}=2\Gamma'(1)-\ln xy,$$

$$\beta(x,y)=L^{-1}\{\frac{2pq}{p^2q^2-1}\}=\frac{1}{2}\{I_0(2\sqrt{xy})-J_0(2\sqrt{xy})\}.$$

And consequently

$$f(x,y)=\frac{1}{2}\int_0^y(\int_0^x(2\Gamma'(1)-\ln(x-\eta)(y-\xi))(I_0(2\sqrt{\eta\xi})-J_0(2\sqrt{\mu\xi}))d\eta)d\xi.$$

In special case x = y = 1, one gets

$$\int_0^{+\infty}\sin\sqrt{t}\sin\frac{1}{\sqrt{t}}\frac{dt}{2t}=f(1,1)=\frac{1}{2}\int_0^1(\int_0^1(2\Gamma'(1)-\ln(1-\eta)(1-\xi))(I_0(2\sqrt{\eta\xi})-J_0(2\sqrt{\mu\xi}))d\eta)d\xi.$$

Application in Solving Fractional PDEs

4.1. Preliminaries:

Lemma.4.1.1: The following relationship holds true

$$\int_0^\infty\frac{e^{-a\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}I_0(bx)x\,dx=\frac{e^{-y\sqrt{a^2-b^2}}}{\sqrt{a^2-b^2}},$$

where $a>b>0, y>0.$

Proof: From the integral representation of K_μ in the form of

$$\frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}}=\frac{a^\mu}{2^{\mu+1}}\int_0^\infty e^{-t-(\frac{a^2(x^2+y^2)}{4t})}\frac{dt}{t^{\mu+1}},$$

We get the following relationship

$$\int_0^\infty\frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}}I_0(bx)x\,dx=\frac{a^\mu}{2^{\mu+1}}\int_0^\infty I_0(bx)x\int_0^\infty e^{-t-(\frac{a^2(x^2+y^2)}{4t})}\frac{dt}{t^{\mu+1}}dx,$$

changing the order of integration, we obtain

$$\frac{1}{(2a)^\mu}(a^2-b^2)^{\mu-1}\int_0^\infty e^{-u-(\frac{y^2(a^2-b^2)}{4u})}\frac{du}{u^\mu}=\frac{1}{a^\mu}\left(\frac{\sqrt{a^2-b^2}}{y}\right)^{\mu-1}K_{\mu-1}(y\sqrt{a^2-b^2}).$$

Setting $\mu=\frac{1}{2}$, and using the suitable relation for $K_{\frac{1}{2}},K_{\frac{-1}{2}}$ in terms of modified Bessel functions of order zero I_0, , one gets

$$\int_0^\infty\frac{e^{-a\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}I_0(bx)x\,dx=\frac{e^{-y\sqrt{a^2-b^2}}}{\sqrt{a^2-b^2}}.$$

Lemma.4.1.2 (Bobylev-Cercignani): Let F (p) is an analytic function having no singularities in the cut plane $C \setminus R_-$. Assume that $\overline{F(p)} = F(\overline{p})$ and the limiting value

$$F^{\pm}(t) = \lim_{\phi \to \pi^-} F(te^{\pm i\phi}), \quad F^+(t) = \overline{F^-(t)}$$

exist for almost all t>0. Let

(i) $F(p) = o(1)$ for $|p| \to \infty$ and $F(p) = o(|p|^{-1})$ for $|p| \to 0$, uniformly in any sector $|\arg p| < \pi - \eta, \pi > \eta > 0$;

(ii) There exists $\varepsilon > 0$ such that for every $\pi - \varepsilon < \phi \le \pi$,

$$\frac{F(re^{\pm i\phi})}{1+r} \in L^1(R_+), \quad |F(re^{\pm i\phi})| \le a(r),$$

Where a(r) does not depend on ϕ and $a(r)e^{-\delta r} \in L^1(R_+)$ for any $\delta > 0$. Then in the notation of the problem,

$$L^{-1}[F] = \frac{1}{\pi}\int_0^\infty \operatorname{Im}[F^-(\eta)]e^{-t\eta} d\eta.$$

Proof: See (Bobylev 2002).

Corollary: In the above lemma, let $F(s) = e^{-as^\beta}$, with $a > 0, \beta > 0$, then F(s) will satisfy the conditions of lemma. We check some of the conditions of lemma as following:
The limiting value

$$F^-(\eta) = \lim_{\phi \to -\pi} F(\eta e^{-i\phi}) = \lim_{\phi \to -\pi} (e^{-a\eta^\beta e^{-i\beta\phi}}) = e^{-a\eta^\beta e^{i\beta\pi}} = e^{-a\eta^\beta(\cos\beta\pi + i\sin\beta\pi)},$$

exists because a>0 , β>0 and sinβπ and cosβπ are bounded. F(s) satisfies the other conditions as well. Therefore

$$\operatorname{Im}(F(\eta e^{i\pi})) = -e^{-a\eta^\beta \cos\beta\pi}\left\{\sin(a\eta^\beta \sin\beta\pi)\right\}.$$

We apply the above lemma to get inverse of $F(s)$ in the form

$$f(t) = \frac{1}{\pi}\int_0^\infty e^{-t\eta - a\eta^\beta \cos\beta\pi} \sin\{(a\sin\beta\pi)\eta^\beta\} d\eta.$$

Special case: $a = 1, \beta = 0.5;$

$$f(t) = \frac{1}{\pi}\int_0^\infty e^{-t\eta} \sin\sqrt{\eta}\, d\eta,$$

making a change of variable $\eta = u^2$ and integrating by parts, we obtain

$$f(t) = \frac{1}{\pi t}\int_0^\infty e^{-tu^2} \cos u\, du, \tag{4.1.1}$$

By using table of integrals, we get

$$f(t) = \frac{e^{-\frac{1}{4t}}}{2t\sqrt{\pi t}}.$$

Lemma.4.1.3: Assume that

$$G(s) = L\{g(t); t \to s\},$$

Then

$$L^{-1}\{\frac{1}{s}G(\frac{1}{s}); s \to t\} = \int_0^\infty J_0(2\sqrt{t\eta})g(\eta)d\eta.$$

Proof: See (Duffy 2004)

4.2 Main results:

In this section, the authors consider boundary value problems for certain time fractional partial differential equations. In this work, only Laplace transformation is considered as a powerful tool to solve the above mentioned problems. This goal has been achieved by formally deriving exact analytical solution.

Problem 4.2.1: Consider a semi finite string vibrations with friction described by the fractional equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = a^2 \frac{\partial^2 u}{\partial x^2} + b\frac{\partial^\alpha u}{\partial t^\alpha} + \lambda u, \qquad 0.5 \le \alpha \le 1, \qquad (4.2.1)$$

in which a, b are positive constants, with initial and boundary conditions in the form

$$u(0,x) = u_t(0,x) = 0, \quad u_x(t,0) = \sum_{k=1}^N V_k \delta(t - T_k), \quad \lim_{x\to\infty} u(t,x) < \infty.$$

Solution: Let us assume that

$$U(p,q) = L_2\{u(t,x); t \to p, x \to q\}.$$

We have

$$L_2\{\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}\} = p^{2\alpha}U(p,q), \quad L_2\{\frac{\partial^2 u}{\partial x^2}\} = q^2 U(p,q) - qH(p) - \sum_{k=1}^N V_k e^{-T_k p},$$

$$L_2\{\frac{\partial^\alpha u}{\partial t^\alpha}\} = p^\alpha U(p,q),$$

in which $H(p) = L\{u(x,0); x \to p\}$.

Substituting in (4.2.1) we get

$$U(p,q) = \frac{a^2(qH(p) + \sum_{k=1}^N V_k e^{-T_k p})}{a^2 q^2 - (p^{2\alpha} - bp^\alpha - \lambda)}. \qquad (4.2.2)$$

On the other hand the roots of denominator must satisfy the numerator as well

$$H(p) = -\frac{a\sum_{k=1}^N V_k e^{-T_k p}}{\sqrt{p^{2\alpha} - bp^\alpha - \lambda}}.$$

Now substituting in (4.2.2) we get

$$U(p,q)=\frac{a^2q\left(-\frac{a\sum_{k=1}^{N}V_k e^{-T_k p}}{\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}\right)+a^2\sum_{k=1}^{N}V_k e^{-T_k p}}{a^2q^2-(p^{2\alpha}-bp^{\alpha}-\lambda)},$$

We can rewrite above equation in the form

$$U(p,q)=-\frac{a^2\sum_{k=1}^{N}V_k e^{-T_k p}}{\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}\,(aq+\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda})}.$$

Now we should invert the above equation. Let us invert with respect to q first by using residue theorem

$$U(p,x)=-\frac{a\sum_{k=1}^{N}V_k e^{-T_k p}}{\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}\exp(-a^{-1}x\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}).$$

Now we should invert the above relationship w.r.t p

$$u(t,x)=-a\sum_{k=1}^{N}V_k\left(L^{-1}\left\{\frac{e^{-a^{-1}x\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}}{\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}e^{-T_k p};p\to t\right\}\right),$$

on the other hand $L^{-1}\left\{F(p)e^{-T_k p}\right\}=f(t-T_k)$, so it suffices to evaluate

$$h(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-\frac{x}{a}\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}}{\sqrt{p^{2\alpha}-bp^{\alpha}-\lambda}}e^{pt}dp.$$

First, let $\alpha=1$;

$$U_1(p,x)=\frac{e^{-\frac{x\sqrt{\lambda+b^2/4}}{a}\sqrt{(p-b/2)^2-1}}}{\sqrt{\lambda+b^2/4}\,(\sqrt{(p-b/2)^2-1})},$$

in which $\lambda+\frac{b^2}{4}>0.$

Now from lemma.4.1.1 and letting $y=\frac{x}{a}\sqrt{\lambda+\frac{b^2}{4}},a=p-\frac{b}{2},b=i$ we have

$$u_1(t,x)=L^{-1}\{U_1(p,x);p\to t\}=\frac{1}{\sqrt{\lambda+\frac{b^2}{4}}}I_0(\sqrt{t^2-\frac{x^2}{a^2}(\lambda+\frac{b^2}{4})}).$$

Now by lemma 2.3 we have

$$L^{-1}\{F(p);p\to t\}=\frac{1}{2t\sqrt{\pi t(\lambda+\frac{b^2}{4})}}\times\int_0^{\infty}\tau I_0(\sqrt{\tau^2-\frac{x^2}{a^2}(\lambda+\frac{b^2}{4})})e^{-\frac{\tau^2}{4t}}d\tau.$$

Research Article

by the fact that $L^{-1}\left\{F(p)e^{-T_k p}\right\} = f\ (t - T_k)$, the final solution will be obtained as bellow

$$u(t,x) = \frac{1}{2\sqrt{\pi(t-T_k)^3(\lambda+\frac{b^2}{4})}} \times \int_0^\infty \tau I_0(\sqrt{\tau^2 - \frac{x^2}{a^2}(\lambda+\frac{b^2}{4})})e^{-\frac{\tau^2}{4(t-T_k)}}d\tau.$$

Problem 4.2.2: Let us consider the following time fractional four terms heat equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^2 u}{\partial x^2}) - bu,$$

with the following boundary conditions

$$u(0,t) = \sin\lambda t, \quad u(x,0) = au_{xx}(x,0), \quad \lim_{x\to+\infty}\left|u(x,t)\right| < +\infty.$$

Solution:. Taking Laplace transform of F.P.D.E and B.Cs w.r.t. t to

$$(1+as^\alpha)U_{xx} - (s^\alpha + b)U = 0,$$

with

$$U(0,s) = \frac{\lambda}{s^2+\lambda^2}, \lim_{x\to+\infty}\left|U(x,s)\right| < +\infty,$$

where U is Laplace transform of the function $u(x,t)$ with respect to t.
Consequently

$$U(x,s) = \frac{\lambda}{s^2+\lambda^2}\exp(-x\sqrt{\frac{s^\alpha+b}{1+as^\alpha}}). \qquad (4.2.3)$$

At this point, let us assume that

$$G(s) = \frac{(s^\alpha + \frac{1}{a})\lambda}{s^2+\lambda^2},$$

$$F(s) = \frac{1}{s^\alpha + \frac{1}{a}}\exp(-x\sqrt{\frac{s^\alpha+b}{1+as^\alpha}}). \qquad (4.2.4)$$

Consider the function F1(s) which is defined by letting α=1 in F(s) as following

$$F_1(s) = \frac{1}{s+\frac{1}{a}}\exp(-\frac{x}{\sqrt{a}}\sqrt{1+\frac{ab-1}{as+1}}).$$

Assume $ab \geq 1$, then let

$$ab - 1 = k^2, \frac{x}{\sqrt{a}} = m$$

,so we have

$$F_1(s) = \frac{1}{s+\frac{1}{a}}\exp(-m\sqrt{1+\frac{k^2}{as+1}}),$$

by using inversion formula we get

$$f_1(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s+\frac{1}{a}}e^{-m\sqrt{1+\frac{k^2}{as+1}}+st}ds.$$

By making a linear change of variable $as+1=p, ds=\frac{dp}{a}$ the above equation takes the form

$$f_1(x,t)=e^{-\frac{t}{a}}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{1}{p}e^{-m\sqrt{1+k^2(\frac{1}{p})}+\frac{p}{a}t}dp\right).$$

Let $p=aq$ then we have

$$f_1(x,t)=e^{-\frac{t}{a}}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{1}{q}e^{-m\sqrt{1+\frac{k^2}{a}q}+qt}dq\right).$$

From the following inversion formula (see tables of inverse of Laplace transform)

$$e^{-m\sqrt{1+\frac{k^2}{a}p}}\doteq\frac{amt^{-\frac{3}{2}}}{2k^2\sqrt{\pi}}e^{-(\frac{at}{k^2}+\frac{m^2}{4t})},$$

and lemma 4.1.3 we obtain

$$f_1(x,t)=\frac{ame^{-\frac{t}{a}}}{2k^2\sqrt{\pi}}\left(\int_0^\infty J_0(2\sqrt{t\eta})\eta^{-3/2}e^{-(\frac{a\eta}{k^2}+\frac{m^2}{4\eta})}d\eta\right).$$

From lemma 2.2 and the corollary of lemma 4.1.2 we get

$$f(x,t)=\int_0^\infty f_1(x,\tau)\left[\frac{1}{\pi}\int_0^\infty \exp(-t\eta-\tau\eta^\alpha\cos\alpha\pi)\sin(\tau\sin\alpha\pi)\eta^\alpha d\eta\right]d\tau. \qquad (4.2.5)$$

On the other hand the inverse of G(s) in (4.2.4)) will be obtained as bellow

$$G(s)=s^\alpha\frac{\lambda}{s^2+\lambda^2}+\frac{1}{a}.\frac{\lambda}{s^2+\lambda^2}=s^\alpha L\{\sin(\lambda t)\}-\sin 0+\frac{1}{a}L\{\sin(\lambda t)\},$$

Therefore we have

$$g(t)=\frac{\partial^\alpha}{\partial t^\alpha}\sin(\lambda t)+\frac{1}{a}\sin(\lambda t), \qquad (4.2.6)$$

In which the fractional derivative is in Caputo sense, which is defined as

$${}_aD_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,$$

In which n is the smallest integer larger than α. Now, from (4.2.5),(4.2.6) and convolution theorem, the final solution is obtained as following

$$u(x,t)=g(t)*f(x,t)$$

$$=\int_0^\infty\left\{\left[\int_0^\infty f_1(x,\tau)\left[\frac{1}{\pi}\int_0^\infty\exp(-\mu\eta-\tau\eta^\alpha\cos\alpha\pi)\sin(\tau\sin\alpha\pi)\eta^\alpha d\eta\right]d\tau\right]\right\}\times$$

$$\times\left(\frac{\partial^\alpha}{\partial(t-\mu)^\alpha}\sin(\lambda(t-\mu))+\frac{1}{a}\sin(\lambda(t-\mu))\right)d\mu.$$

CONCLUSION

The paper is devoted to study and application of multi – dimensional Laplace transforms for some special combinations of functions of scalar variables. The multi-dimensional Laplace Transform provides powerful method for analyzing linear systems.

The other type of application is oriented for finding analytic solution of the time fractional wave and heat equation using Laplace transform. The time – fractional is considered in the Caputo sense. It may be concluded that the method is very powerful and efficient technique for solving the model.

Finally, the recent appearance of time - fractional heat equation as models in some fields such as the thermal diffusion in fractal media makes it necessary to investigate the method.

REFERENCES

Aghili A and Salkhordeh Moghaddam B (2004). Laplace transforms pairs of n- dimensions and a Wave equation. *International Journal of Mathematics* **5(**4) 377-382.

Aghili A and Salkhordeh Moghaddam B (2006). Multi-Dimensional Laplace Transform and systems of partial differential equations. *International Journal of Mathematics* **1(**6) 21-24.

Aghili A and Salkhordeh Moghaddam B (2008). Laplace transforms pairs of N- dimensions and second order linear differential equations with constant coefficients. *Annales Mathematicae et informaticae* **35** 3-10.

Bobylev AV and Cercignani C (2002). The inverse Laplace transform of some analytic functions with an application to the eternal solutions of the Boltzmann equation. *Applied Mathematics Letters* 15 807-813.

Dahiya RS and Saberi Nadjafi J (1999). Theorems on N-Dimensional Laplace Transforms and their applications. *15th Annual Conference of Applied Mathematics, University of Central Oklahoma*. *Electronic Journal of Differential Equations* Conference **2** 61-74.

Dahiya RS and Vinayagamoorthy M (1990). Laplace Transform pairs of n-dimensions and heat conduction problem. *Mathematical and Computer Modelling* **13(**10) 35-50.

Ditkin VA and Prudnikov AP (1979). Calcul operationnel,Edition de Moscou,

Duffy DG (2004). Transform methods for solving partial differential equations, Chapman and Hall/CRC NewYork.

Kilbas A and Trujillo J (2002). Differential equation of fractional order: methods, results and problems. *II Applied Analysis* **81**(2) 435-493.