International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2012 Vol. 2 (4) October-December, pp.5-10/Srinivas et al.
Research Article

RADIAL VIBRATIONS IN A MICRO-ISOTROPIC, MICRO-ELASTIC SOLID SPHERE

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ABSTRACT
The frequency equations are derived for the radial vibrations in a micro-isotropic, micro-elastic solid sphere. It is interesting to observe that two additional frequencies are found which are not encountered in classical theory of elasticity. The result of the classical case is obtained as a particular case of it.

Key Words: *Radial Vibrations, Micro-Elastic Solid Sphere*

INTRODUCTION
Modern engineering structures are often made up of material possessing of internal structure. Polycrystalline materials, materials with fibrous and coarse grain structures come under this category. In such materials the ratio Δm/Δv varies in the range Δv<Δv* where Δv* is critical volume (Eringen, 1968). Therefore, the classical elasticity is inadequate to represent the behavior of such materials. The inadequacy of classical continuum approach has lead to the development of the theory of micro-continua. The general theory of micro orphic materials introduced by Eringen and Suhubi (1966a and 1964a) deal with substances which exhibit certain microscopic effects arising from the local structure of media. Eringen (1968) derived a theory of micropolar elasticity; the model of micropolar elasticity is quite adequate for the representation of solids composed of rigid solid cylinders or dum-bell type molecules. By imposing the restriction on the components of the position of point in the micro element relative to the centre of mass are linear, Eringen and Suhubi linearized the theory and named it as theory of micromorphic materials. To analyze the mechanical behavior of the micromorphic solid, there will be 12 second order partial differential equations in 12 unknowns and 18 elastic moduli. Koh (1970) developed a somewhat simpler theory by extending the concept of coincidence of the principal directions of stress and strain in classical elasticity to the micro-elastic solid. Imposing a particular form of micro-isotropy, Koh obtained special constraints on the elastic moduli, thereby reducing the number from 18 to 10 in the special case. The theory developed by Koh is known as micro-isotropic, micro-elastic solid. Sato and Usami (1962a); Sato *et al.*, (1962, 1963) and Usami and Sato (1964) studied the propagation of various types of disturbances in a sphere. The problem of wave reflection from surface of the sphere was considered by Nomura and Takaku (1964). Radial vibrations of an isotropic elastic sphere and other problems are given in the book by Ghosh (1975).
In this paper, the problem of radial vibrations in a micro-isotropic, micro-elastic solid sphere is studied. It is interesting to observe that two additional frequencies are obtained, which are not encountered in the classical theory of elasticity. Further, the result of classical case is obtained as a particular case of this paper.

BASIC EQUATIONS
The equations of motion and the constitute equations of micro-isotropic, micro-elastic solid without body forces and body couples are given by Parameshwaran and Koh (1973)
The displacement equations of motion are

$$(A_1 + A_2 - A_3)u_{p,pm} + (A_2 + A_3)u_{m,pp} + 2A_3\varepsilon_{pkm}\phi_{p,k} = \rho\frac{\partial^2 u_m}{\partial t^2} \qquad (1)$$

$$2B_3\phi_{p,mm} + 2(B_4 + B_5)\phi_{m,mp} - 4A_3(r_p + \phi_p) = \rho j\frac{\partial^2 \phi_p}{\partial t^2} \qquad (2)$$

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$$B_1\phi_{pp,kk}\delta_{ij} + 2B_2\phi_{(ij),kk} - A_4\phi_{pp}\delta_{ij} - 2A_5(\phi_{ij}) = \frac{1}{2}\rho j\frac{\partial^2\phi_{(ij)}}{\partial t^2} \quad (3)$$

where

$$A_1 = \lambda + \sigma_1, \qquad B_1 = \tau_3,$$
$$A_2 = \mu + \sigma_2, \qquad 2B_2 = \tau_7 + \tau_{10},$$
$$A_3 = \sigma_5, \qquad B_3 = 2\tau_4 + 2\tau_9 + \tau_7 - \tau_{10},$$
$$A_4 = -\sigma_1, \qquad B_4 = -2\tau_4,$$
$$A_5 = -\sigma_2, \qquad B_5 = -2\tau_9 \quad (4)$$

and

$$3A_1 + 2A_2 > 0, \qquad A_2 > 0, \qquad A_3 > 0,$$
$$3A_4 + 2A_5 > 0, \qquad A_5 > 0,$$
$$2B_1 + 2B_2 > 0, \qquad B_2 > 0,$$
$$B_5 > 0, \qquad -B_2 < B_4 < B_2, \qquad B_3 + B_4 + B_5 > 0 \quad (5)$$

The stress, couple-stress and stress moment are as follows.

$$t_{(km)} = A_1 e_{pp}\delta_{km} + 2A_2 e_{km} \quad (6)$$

$$t_{[km]} = \sigma_{[km]} = 2A_3\varepsilon_{pkm(}r_p + \phi_p) \quad (7)$$

$$\sigma_{(km)} = -A_4\phi_{pp}\delta_{km} - 2A_5\phi_{(km)} \quad (8)$$

$$t_{k(mn)} = B_1\phi_{pp,k}\delta_{mn} + 2B_2\phi_{(m,n),k} \quad (9)$$

$$m_{kl} = -2(B_3\phi_{l,k} + B_4\phi_{k,l} + B_5\phi_{p,p}\delta_{kl}) \quad (10)$$

where ρ is the average mass density, j is the micro-inertia. The macro displacement in the micro elastic continuum is denoted by u_k and the micro deformation by ϕ_{mn} for the linear theory, we have the macro strain $e_{km} = e_{(k,m)}$, the macro rotation vector $r_k = \frac{1}{2}\varepsilon_{kmn}u_{n,m}$, the micro-strain $\phi_{(m,n)}$ and micro-rotation $\phi_p = \frac{1}{2}\varepsilon_{pkm}\phi_{km}$.The stress measures are the asymmetric stress (macro-stress) t_{kmn}, the relative stress (micro-stress) σ_{km} and the stress moment t_{kmn}. Also the couple stress tensor $m_{kp}\varepsilon_{pnm}t_{kmn}$. The symbol ( ) shows that the quantity is symmetric and [ ] shows the quantity is skew-symmetric. λ, μ, σ_1, σ_2, σ_5, τ_3, τ_4, τ_7, τ_9 and τ_{10} are the ten elastic moduli.

FORMULATION AND SOLUTION OF THE PROBLEM

We consider a micro – isotopic, micro – elastic solid sphere of radius a. we choose the centre of the sphere as the origin of the coordinate system (r,θ,q). If the displacement filed in an elastic medium manifests a radial symmetry with respect to a point that is assumed to be the origin, the radial displacement $\vec{u}$, the radial micro – rotation $\vec{\phi}$ and the radial micro – strain ϕ_{rr} depends only on the radial distance r form the origin and time, and the other components u_θ, u_q, ϕ_θ and ϕ_q are zero. Hence we take

$$\vec{u} = u(r,t)e_r \quad (11)$$

$\vec{\phi} = \phi(r,t)e_r$ (12)

$\phi_{rr} = \phi_{rr}(r,t)$ (13)

where e_r is the unit vector at the position vector in the direction of the tangent to the r – curve.
Under the absence of body forces and body couples the equations of motion (1) to (3) would reduce to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u = \frac{\rho}{(A_1 + 2A_2)}\frac{\partial^2 u}{\partial t^2} \tag{14}$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} - \frac{2}{r^2}\phi - \frac{2A_3}{(B_3 + B_4 + B_5)} = \frac{\rho j}{2(B_3 + B_4 + B_5)}\frac{\partial^2 \phi}{\partial t^2} \tag{15}$$

$$B_1\nabla^2\phi_{rr} + 2B_2\nabla^2\phi_{rr} - A_4\phi_{rr} - 2A_5\phi_{rr} = \frac{\rho j}{2}\frac{\partial^2 \phi_{rr}}{\partial t^2} \tag{16}$$

$$B_1\nabla^2\phi_{rr} - A_4\phi_{rr} = 0 \tag{17}$$

In view of (17) the equation (16) reduces to

$$2B_2\nabla^2\phi_{rr} - 2A_5\phi_{rr} = \frac{\rho j}{2}\frac{\partial^2 \phi_{rr}}{\partial t^2} \tag{18}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$

The boundary conditions are

$$t_{rr} = (A_1+2A_2)\frac{\partial u}{\partial r} + \frac{2A_1}{r}u = 0 \quad \text{at } r = a \tag{19}$$

$$m_{rr} = -2(B_3+B_4+B_5)\frac{\partial \phi}{\partial r} - \frac{4B_5}{r}\phi = 0 \quad \text{at } r = a \tag{20}$$

$$t_{r(rr)} = (B_1+2B_2)\frac{\partial \phi_{rr}}{\partial r} \quad \text{at } r = a \tag{21}$$

We seek the solution of (14) in the form of

$$u(r,t) = F(r)e^{iwt} \tag{22}$$

Substituting (22) in (14) we get

$$\frac{\partial^2 F}{dr^2} + \frac{2}{r}\frac{dF}{dr} - \frac{2}{r^2}F + \frac{w^2\rho}{A_1 + 2A_2}F = 0 \tag{23}$$

Suppose $x = hr$ (24)

where $h^2 = \frac{w^2\rho}{A_1 + 2A_2}$ (25)

Using (24) and (25), the equation (23) can be expressed as

$$\frac{d^2 F}{dx^2} + \frac{2}{x}\frac{dF}{dx} - \frac{2}{x^2}F + F = 0 \tag{26}$$

The general solution of (26) is

$$F(x) = A\frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

Hence, $u(r,t) = A\frac{d}{dx}\left(\frac{\sin x}{x}\right)e^{iwt}$ (27)

where x in terms of r is given by (24) and A is a constant

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Substituting (27) in the boundary condition (19) we get

$$\frac{\tanh a}{ha} = \frac{4A_2}{4A_2 - h^2a^2(A_1 + 2A_2)} \tag{28}$$

which is the periodic equation corresponding to macro displacement. The frequency equation of classical result can be obtained as a particular case of it by allowing σ_1 and σ_2 tends to 0.
Now we seek the solution of (15) in the form

$$\phi = G(r)\, e^{i\omega t} \tag{29}$$

Substituting (29) in (15) we get

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} - \frac{2}{r^2}G - \frac{2A_3}{(B_3 + B_4 + B_5)}G = \frac{-\omega^2\rho j}{2(B_3 + B_4 + B_5)}G \tag{30}$$

This can be written as

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} - \frac{2}{r^2}G + h_1^2 G = 0 \tag{31}$$

where $h_1^2 = \dfrac{\omega^2\rho j - 4A_3}{2(B_3 + B_4 + B_5)}$ (32)

Suppose $y = h_1 r$ (33)

Using (33) the equation (31) can be written as

$$\frac{d^2G}{dy^2} + \frac{2}{y}\frac{dG}{dy} - \frac{2}{y^2}G + G = 0 \tag{34}$$

The general solution of (34) is

$$G(y) = B\frac{d}{dy}\left(\frac{\sin y}{y}\right)$$

Hence, $\phi(r, t) = B\dfrac{d}{dy}\left(\dfrac{\sin y}{y}\right)e^{i\omega t}$ (35)

where y interns of r is given by (33) and B is a constant.
Substituting (35) in the boundary condition (20) we get

$$\frac{\tanh_1 a}{h_1 a} = \frac{2B_3 + 2B_4}{(2B_3 + 2B_4) - h_1^2a^2(B_3 + B_4 + B_5)} \tag{36}$$

which is the frequency equation corresponding to micro rotation It is dispersive in nature as it depends on the frequency. It involves elastic constants other than the classical constants λ and μ. Hence it is an additional wave which is not encountered in the classical elasticity.
Now we seek the solution of (18) in the form of

$$\phi_{rr} = H(r)e^{i\omega t} \tag{37}$$

Substituting (37) in (18) we get

$$2B_2\left[\frac{d^2H}{dr^2} + \frac{2}{r}\frac{dH}{dr}\right] - 2A_5 H + \frac{w^2\rho j}{2}H = 0$$

This can be written as

$$\frac{d^2H}{dr^2} + \frac{2}{r}\frac{dH}{dr} - l^2H = 0 \tag{38}$$

Where $l^2 = \dfrac{4A_5 - \omega^2\rho j}{4B_2}$ (39)

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Let $H(r) = \frac{1}{\sqrt{r}} T(r)$ (40)

Substituting (40) in (38) we get

$$r^2 T^{11}(r) + r T^1(r) - \left(l^2 r^2 + \frac{1}{4}\right) T(r) = 0$$

which can be expressed as

$$r^2 T^{11} + rT^1 + \left[(il)^2 r^2 - \left(\frac{1}{2}\right)^2\right] T = 0$$

It is a Bessel equation, whose solution is

$$T(r) = L_1 J_{\frac{1}{2}}(ilr) + L_2 Y_{\frac{1}{2}}(ilr) \quad (41)$$

where $J_{\frac{1}{2}}, Y_{\frac{1}{2}}$ are Bessel functions with imaginary arguments and is written as

$$T(r) = L_1 I_{\frac{1}{2}}(lr) + L_2 K_{\frac{1}{2}}(lr) \quad (42)$$

where L_1, L_2 are arbitrary constants

Substituting (42) in (40) we get

$$H(r) = \frac{1}{\sqrt{r}}[L_1 I_{\frac{1}{2}}(lr) + L_2 K_{\frac{1}{2}}(lr)]$$

Hence, $\phi_{rr}(r,t) = \frac{1}{\sqrt{r}}[L_1 I_{\frac{1}{2}}(lr) + L_2 K_{\frac{1}{2}}(lr)]e^{iwt}$ (43)

As $r \to \infty, \phi_{rr} \to 0,$ which is possible only if $L_1 = 0$.

For large values of z we have $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$

Therefore,

$$\phi_{rr}(r,t) = L_2 \sqrt{\frac{\pi}{2l}} e^{-lr} \frac{1}{r} e^{iwt} \quad (44)$$

Substituting (44) in the boundary condition (21) we get

$$w^2 = \frac{1}{\rho j}(4A_5 - \frac{4B_2}{a^2}) \quad (45)$$

NUMERICAL CALCULATIONS

The frequency equation (28) can be reduced to

$$\frac{\tan\sqrt{\frac{m_2}{2+m_1}}cx}{\sqrt{\frac{m_2}{2+m_1}}cx} - \frac{4}{4 - c^2 m_2 x^2} = 0 \quad (46)$$

where $\frac{A_1}{A_2} = m_1$, $\frac{\rho}{A_2} = m_2$, $\frac{\omega}{k} = c$, $\frac{2\pi a}{l} = x$ and l is wave length.

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Assuming the non-dimensional quantities $m_1=0.1$ and $m_2=0.3$. We computed the values of phase velocities for various values of $\frac{2\pi a}{l}$ and they are shown graphically in **Figure (1)**

.The frequency equation (36) can be reduced to

$$\frac{\tan\left(as\sqrt{x^2-4}\right)}{\left(as\sqrt{x^2-4}\right)}-\frac{m}{m-s_1a^2(x^2-4)}=0 \qquad (47)$$

where $s=\sqrt{\frac{s_1}{2s_4}}$, $s_1=\frac{A_3}{B_3}$, $s_2=\frac{B_4}{B_3}$, $s_3=\frac{B_5}{B_3}$, $s_4=1+s_2+s_3$, $x=\frac{\omega}{c}$, $c=\sqrt{\frac{A_3}{\rho j}}$, $m=4(1+s_2)$

Assuming the non-dimensional quantities $s_1=1.2$, $s_2=1.5$ and $s_3=1.4$. We computed the values of x for various values of radius a and they are shown graphically in **Figure (2).**It is observed that the proportional phase velocity is always lying between 2 and 2.5 for different values of the radius of sphere a.

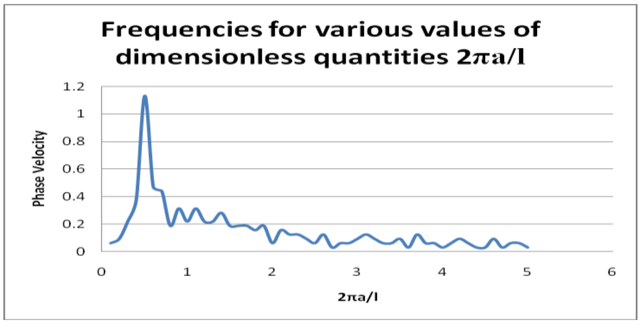

Figure (1)

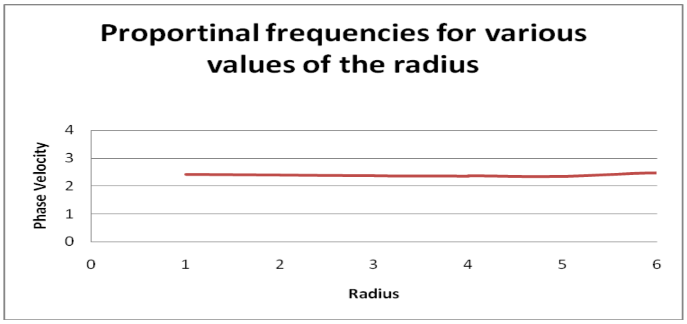

Figure (2)

REFERENCES

Eringen AC (1968). *Theory of micro polar elasticity in fracture*-II**.** Academic Press, New York.

Eringen AC (1966a). *Journal of Mathematics and Mechanics* **15** 909-924.

Eringen AC and Suhubi ES (1964a). Non-linear theory of simple microelastic solids-I. *International Journal of Engineering Science* **2** 189-203.

Ghosh PK (1975). *The mathematics of waves and vibrations.* The Mac Millon Company of India limited, India.

Koh SL (1970). A special theory of microelasticity. *International Journal of Engineering Science* **8**(7) 583-593.

Nomura Y and Takuku K (1964). On the propagation of elastic waves in an isotropic homogeneous sphere. *Journal of the Physical Society of Japan* **7** 204.

Parameshwaran S and Koh SL (1973). Wave propagation in a micro-isotropic, micro-elastic solid. *International Journal of Engineering Science* **2** 95-107.

Sato Y and Usami T (1962a). Basic study on the oscillation of a homogenous elastic sphere I, frequency of free oscillations. *Geophysics Magazine* **31** 15.

Sato Y, Usami T and Ewing M (1962). Basic study on the oscillation of a homogeneous elastic sphere IV, propagation of disturbances on the sphere. *Geophysics Magazine* **31** 237.

Sato Y, Usami T and Ewing M (1963). Basic study on the oscillation of a homogeneous elastic sphere V. *Geophysical Journal of the Royal Astronomy Society* **8** 44.

Usami T and Sato Y (1964). Propagation of spheroid disturbances on a homogenous elastic sphere. *Bulletin of the Earthquake Research Institute*, *Tokyo University* **42** 273.