International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2012 Vol. 2 (3) July-September pp.38-41/Selvi and Thangavelu

Research Article

ON SLIGHTLY P-CONTINUOUS MULTIFUNCTIONS

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ABSTRACT
The concepts of p-continuity between topological spaces are introduced by Thangavelu and Rao. The purpose of this paper is to introduce upper and lower slightly p-continuous multifunctions and investigate their relationships with other multifunctions.

INTRODUCTION AND PRELIMINARIES
In this chapter the concepts of slightly p-continuous and slightly q-continuous multifunctions are introduced and their properties are investigated.

Definitions
Definition 1.1:
Let A be a subset of a topological space X, then A is called
- a p-set[2] if $cl(int(A)) \subseteq int(cl(A))$
- α-open[1] if $A \subseteq int(cl(int(A)))$ and α-closed if $cl(int(cl(A))) \subseteq A$

Definition 1.2:
A multifunction F☹X, τ) → (Y, σ) is said to be upper p-continuous on X if for every open set $x \in X$, for every V in Y containing F(x) there is a p-set U in X such that $x \in U$ and $F(U) \subseteq V$.
Definition 1.3:
A multifunction F: (X, τ) → (Y, σ) is said to be lower p-continuous on X if for every $x \in X$ if for every open set V in Y with $V \cap F(x) \neq \emptyset$ there is a p-set U such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for every $u \in U$.
Definition 1.4:
A multifunction F: X → Y is said to be weakly injective if for any two subsets A_1 and A_2 of X, $F(A_1) \cap F(A_2) \neq \emptyset \Rightarrow A_1 \cap A_2 \neq \emptyset$. A multifunction F: X → Y is said to be weakly bijective if it is both weakly injective and weakly surjective.
Lemma 1.5:
Let a multifunction F: X → Y is weakly surjective. Then for any subset A of X $Y \backslash F(A) \subseteq F(X \backslash A)$.
Proof: Let $y \in Y \backslash F(A)$. Then $y \in Y$ and $y \notin F(x)$ for every $x \in A$. Since F is weakly surjective there is a subset B of X such that F (B) =Y that implies $y \in F(b)$ for some $b \in B$. If $b \in A$ then $F(b) \subseteq F(A)$ so that $y \in F(A)$ contradicting $y \in Y \backslash F(A)$.Therefore b cannot lie in A. Thus $b \in B \backslash A$. This shows that $Y \backslash F(A) \subseteq F(B \backslash A) \subseteq F(X \backslash A)$.
Lemma 1.6:
Let A multifunction F: X → Y is weakly injective. Then for any subset A of X $Y \backslash F(A) \supseteq F(X \backslash A)$.Moreover if F: X → Y is weakly bijective then for any subset A of X, $Y \backslash F(A) = F(X \backslash A)$.
Proof: Let $y \in F(X \backslash A)$. Then $y \in F(x)$ for some $x \in X \backslash A$ Clearly $y \in Y$. Suppose $y \in F(A)$. Then $y \in F(a)$ for some $a \in A$. Therefore $F(x) \cap F(a) \neq \emptyset$. Since F is weakly injective, by Definition 1.4 $\{x\} \cap \{a\} \neq \emptyset$ that implies x=a. This is absurd as $x \notin A$ and $a \in A$. This proves $Y \backslash F(A) \supseteq F(X \backslash A)$. Since F is weakly bijective the above arguments together with Lemma 1.5 imply that $Y \backslash F(A) = F(X \backslash A)$.
The following four Lemmas will be useful in sequel.
Lemma 1.7:
A subset B of X is a p-set if and only if X\B is also a p-set. [2]

Lemma 1.8:
A subset B of X is a p-set in (X,τ) if and only if it is p-set in (X,τ^{α}) where τ^{α} is a topology of α-open sets in (X,τ). [2]
Lemma 1.9:
The intersection of a p-set with a clopen set is again a p-set [2].
Lemma 1.10:
Let Y be a clopen set in X and $B\subseteq Y\subseteq X$. Then B is a p-set in X if and only if it is a p-set in Y. [2]

Upper Slightly P-Continuity

In this section upper slightly p-continuous functions are introduced and characterized
Definition 2.1:
A multifunction $F: (X,\tau) \rightarrow (Y, \sigma)$ is said to be upper slightly p-continuous on X if for every $V\in CO(Y, \sigma)$ containing F(x) there is a p-set U such that $x\in U$ and $F(U) \subseteq V$. An upper slightly p-continuous multifunction is upper slightly *m*-continuous if $m_X = p(X,\tau)$.
Theorem 2.3:
A multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ is upper slightly p-continuous on X if and only if for every x and for every clopen set B in Y with $B\cap F(x)=\emptyset$, there is a p-set U in X such that $x\in U$ and $F(U)\cap B =\emptyset$.
Proof: Suppose $F: (X,\tau) \rightarrow (Y,\sigma)$ is upper slightly p-continuous on X. Let $x\in X$ and B be a clopen set in Y such that $B\cap F(x) =\emptyset$. Then $Y\backslash B$ is also clopen in Y with $F(x)\subseteq Y\backslash B$. Using Definition 2.1 there is a p-set U in X such that $x\in U$ and $F(U) \subseteq Y\backslash B$. This shows that $F(U) \cap B =\emptyset$. Conversely we assume that for every x and for every clopen set B in Y with $B\cap F(x) =\emptyset$, there is a p-set U in X such that $x\in U$ and $F(U)\cap B =\emptyset$. Fix $x\in X$ and a clopen set V in Y with $F(x) \subseteq V$. Then $F(x) \cap (Y\backslash V) = \emptyset$. Then by the assumptions there is a p-set U with $x \in U$ and $F(U) \cap (Y\backslash V) = \emptyset$ that implies $F(U) \subseteq V$. By using Definition 2.1 F is upper slightly p-continuous.
Theorem 2.4:
Suppose $F: (X,\tau) \rightarrow (Y,\sigma)$ is weakly bijective. Then F is upper slightly p-continuous on X if and only if for every x and for every clopen set B in Y with $B\cap F(x)=\emptyset$, there is a p-set W in X such that $x\notin W$ and $B\subseteq F(W)$.
Proof: Suppose F is upper slightly p-continuous. Fix $x\in X$. Let B be a clopen set in Y and $B\cap F(x) =\emptyset$. Since F is upper p-continuous by using Theorem 2.3 there is a p-set U with $x \in U$ such that $F(U) \cap B=\emptyset$. This implies that $F(U) \subseteq Y\backslash B$ that implies $Y\backslash F(U)\supseteq B$. Since F is weakly bijective, $F(X\backslash U) = Y\backslash F(U)\supseteq B$. Taking $W = X\backslash U$ and using Lemma 1.7 we see that W is a p-set with $x\notin W$ and $B\subseteq F(W)$. Conversely, let us assume that for every x and for every clopen set B in Y with $B\cap F(x) =\emptyset$, there is a p-set W in X such that $x\notin W$ and $B\subseteq F(W)$. In order to prove that F is upper slightly p-continuous fix $x \in X$ and a clopen set B with $B\cap F(x) =\emptyset$. Then by our assumption there is a p-set W in X such that $x\notin W$ and $B\subseteq F(W)$. Therefore $B\cap (Y\backslash F(W)) =\emptyset$. Since F is weakly bijective, by Lemma 1.6, $Y\backslash F(W) = F(X\backslash W)$ that implies $B\cap F(X\backslash W) =\emptyset$. Then by applying Theorem 2.3, F is upper slightly p-continuous.
Theorem 2.5:
A multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ is upper slightly p-continuous on X if and only if $F: (X, \tau^{\alpha}) \rightarrow (Y, \sigma)$ is upper p-continuous;
Proof: Let $F :(X,\tau) \rightarrow (Y,\sigma)$ be upper p-continuous. Suppose $x \in X$ and V is a clopen set in Y with $F(x)\subseteq V$. Then by using Definition 2.1, there is a p-set U in (X,τ) such that $F(U) \subseteq V$. Again by using Lemma 1.8, U is also a p-set in (X,τ^{α}). This proves that $F: (X,\tau^{\alpha}) \rightarrow (Y,\sigma)$ is upper slightly p-continuous. Conversely we assume that $F: (X,\tau^{\alpha}) \rightarrow (Y,\sigma)$ is upper slightly p-continuous. Let $x \in X$ and V be an open set in Y with $F(x)\subseteq V$. Then by using Definition 1.3, there is p-set U in (X,τ^{α}) such that $F(U) \subseteq V$. Again by using Lemma 1.8, U is also a p-set in (X,τ). This proves that $F: (X,\tau) \rightarrow (Y,\sigma)$ is upper p-continuous. Thus (i) am proved.

The next two Lemmas on p-sets are used to show that the restriction of an upper p-continuous multifunction to a clopen set is upper p-continuous.

Theorem 2.6:

If a multifunction F: $X\rightarrow Y$ is upper slightly p-continuous and X_0 is clopen, then the restriction $F/_{X_0}$: $X_0\rightarrow Y$ is upper p-continuous.

Proof: Suppose a multifunction F: $X\rightarrow Y$ is upper slightly p-continuous on X. Fix $x\in X_0$ and V is a clopen set in Y with $F(x)\subseteq V$. Since F is upper slightly p-continuous on X, by Definition 2.1, there is a p-set U in X such that $F(U)\subseteq V$. Let $U_0 = U\cap X_0$. Then using Lemma 1.9, U_0 is a p-set in X. Again by using Lemma 1.10, U0 is a p-set in X_0. Since $F(U_0)\subseteq F(U)\subseteq V$, by Definition 2.1, $F/_{X_0}$ is upper p-continuous at x and hence on $X_{0.}$

Theorem 2.9:

A multifunction F: $X\rightarrow Y$ is upper slightly p-continuous on X if and only if for every x in X and for every clopen set V in Y with $x\in F^+(V)$ there is a p-set U in X such that $x\in U$ and $U\subseteq F^+(V)$.

Proof: We assume that F is upper slightly p-continuous on X. Let $x\in X$. Let V be a clopen set in Y such that $x\in F^+(V)$. This implies $F(x)\subseteq V$. Then by Definition 2.1 there is a p-set U in X such that $x\in U$ and $F(U)\subseteq V$. Now $u\in U \Rightarrow F(u)\subseteq V \Rightarrow u\in F^+(V)$. This proves that $U\subseteq F^+(V)$. Conversely we assume that for every x in X and for every clopen set V in Y with $x\in F^+(V)$ there is a p-set U in X such that $x\in U$ and $U\subseteq F^+(V)$. Now let $x\in X$ and let V be a clopen set in Y with $F(x)\subseteq V$. Since $F(x)\subseteq V$, $x\in F^+(V)$. By our assumption there exists a p- set U of X such that $x\in U\subseteq F^+(V)$. We claim that $F(U)\subseteq V$. If $u\in U$, then $u\in F^+(V)$, that implies $F(u)\subseteq V$. Therefore $\bigcup_{u\in U} F(u)\subseteq V$ and hence $F(U)\subseteq V$. Therefore by Definition 2.1, F is upper p-continuous at x and hence on X.

Theorem 2.10:

Let X is a space in which the family of all p-sets is closed under arbitrary union. A multifunction F: $X\rightarrow Y$ is upper slightly p-continuous on X if and only if for every clopen set V in Y, $F^+(V)$ is a p-set in Y.

Proof: Suppose F is upper slightly p-continuous on X. Let V be a non empty clopen set in Y. Let $x\in F^+(V)$. Then by Theorem 2.9, there is p-set U in X with $x\in U\subseteq F^+(V)$. Therefore $F^+(V)$ are a union of p-sets. By hypothesis, $F^+(V)$ are a p-set. Conversely we assume that for every clopen set V in Y, $F^+(V)$ is a p-set in X. Let V be a clopen set with $x\in F^+(V)$. Since $F^+(V)$ is a p-set in X, taking $U = F^+(V)$ and using Theorem 2.9, F is upper slightly p-continuous.

Example 2.11:

Let X = Y= {a, b, c, d), and $\tau = \sigma = \{\emptyset, \{b, c, d\}, \{a, b\}, \{c, d\}, \{b\}, X\}$. Define F: $(X, \tau)\rightarrow(X, \sigma)$ by F(a) ={ a, d}, F(b) ={a, c} , F(c) = {c, d} and F(d) = {a, c}. F is upper slightly p-continuous.

Lower Slightly p-Continuity

In this section lower slightly p-continuous functions are introduced and characterized.

Definition 3.1:

A multifunction F: $(X, \tau)\rightarrow(Y, \sigma)$ is said to be lower slightly p-continuous on X if for every clopen set V in Y with $V\cap F(x)\neq\emptyset$ there is a p-set U such that $x\in U$ and $F(u)\cap V\neq\emptyset$ for every $u\in U$.

Theorem 3.2:

Let a multifunction F: $(X,\tau)\rightarrow(Y,\sigma)$ be weakly injective. If F is lower slightly p-continuous on X then for every x and for every clopen set B in Y with $B\cap F(x)=\emptyset$, there is a p-set U in X such that $x\in U$ and $B\cup F(X\backslash u)\neq Y$ for every $u\in U$.

Proof: Suppose F is lowering slightly p-continuous. Fix x in X and a clopen subset B of Y with $B\cap F(x) =\emptyset$. Then $F(x)\subseteq Y\backslash B$. Since $Y\backslash B$ is clopen in Y and since $(Y\backslash B)\cap F(x) = F(x)\neq\emptyset$, by using Definition 3.1, there is a p-set U in X with $x\in U$ and $F(u)\cap V\neq\emptyset$ for every $u\in U$. This implies $(Y\backslash F(u))\cup(Y\backslash V)\neq Y$

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that implies $(Y\backslash F(u)) \cup B \neq Y$. Since F is weakly injective by Lemma 3.2.8, $F(X\backslash u) \subseteq Y\backslash F(u)$, that implies $F(X\backslash u) \cup B \neq Y$ for every $u \in U$.

Theorem 3.3:

A multifunction $F: (X, \tau) \to (Y, \sigma)$ is lower slightly p-continuous on X if and only if $F: (X, \tau^{\alpha}) \to (Y, \sigma)$ is lower slightly p-continuous;

Proof: Let $F: (X,\tau) \to (Y,\sigma)$ be lower slightly p-continuous. Suppose $x \in X$ and V is a clopen set in Y with $F(x) \cap V \neq \emptyset$. Then by using Definition 3.1, there is p-set U in (X, τ) such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for all $u \in U$. Again by using Lemma 1.9, U is also a p-set in (X, τ^{α}). This proves that $F: (X,\tau^{\alpha}) \to (Y,\sigma)$ is lower slightly p-continuous. Conversely we assume that $F: (X,\tau^{\alpha}) \to (Y,\sigma)$ is lower slightly p-continuous. Let $x \in X$ and V be a clopen set in Y with $F(x) \cap V \neq \emptyset$. Then by using Definition 3.1, there is p-set U in (X,τ^{α}) such that $F(u) \cap V \neq \emptyset$ for all $u \in U$. Again by using Lemma 1.9, U is also a p-set in (X,τ). This proves that $F: (X,\tau) \to (Y,\sigma)$ is lower slightly p-continuous.

Theorem 3.4:

If a multifunction $F: X \to Y$ is lowering slightly p-continuous and X_0 is clopen, then the restriction $F/_{X_0}$: $X_0 \to Y$ is lower slightly p-continuous.

Proof: Suppose a multifunction $F: X \to Y$ is lowering slightly p-continuous on X. Fix $x \in X_0$ and V is a clopen set in Y with $F(x) \cap V \neq \emptyset$. Since F is lower slightly p-continuous on X, by Definition 3.1, there is a p-set U in X such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for all $u \in U$. Let $U_0 = U \cap X_0$. Then using Lemma 1.9, U_0 is a p-set in X. Again by using Lemma 1.10, U_0 is a p-set in X_0. Since $U_0 \subseteq U$, it follows that $F(u) \cap V \neq \emptyset$ for all $u \in U_0$. So by Definition 3.1, $F/_{X_0}$ is lower slightly p-continuous on X and hence on X_0.

Theorem 3.7:

A multifunction $F: X \to Y$ is lower slightly p-continuous on X if and only if for every x in X and for every clopen set V in Y with $x \in F^-(V)$ there is a p-set U in X such that $x \in U$ and $U \subseteq F^-(V)$.

Proof: We assume that F is lowering slightly p-continuous on X. Let $x \in X$. Let V be a clopen set in Y such that $x \in F^-(V)$. This implies $F(x) \cap V \neq \emptyset$. Then by Definition 3.1, there is a p-set U in X such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for all $u \in U$. Now $u \in U \Rightarrow F(u) \cap V \neq \emptyset \Rightarrow u \in F^-(V)$. This proves that $U \subseteq F^-(V)$. Conversely we assume that for every x in X and for every clopen set V in Y with $x \in F^-(V)$ there is a p-set U in X such that $x \in U$ and $U \subseteq F^-(V)$. Now let $x \in X$ and let V be a clopen set in Y with $F(x) \cap V \neq \emptyset$. Since $F(x) \cap V \neq \emptyset$, $x \in F^-(V)$. By our assumption there exists a p-set U of X such that $x \in U \subseteq F^-(V)$. If $u \in U$, then $u \in F^-(V)$ that implies $F(u) \cap V \neq \emptyset$ for all $u \in U$. Therefore by Definition 3.1, F is lowering p-continuous on X.

Examples can be constructed to show that the reverse implications are not true.

Example 3.8:

Let X = Y= {a, b, c, d) and $\tau = \sigma = \{\emptyset, \{b, c, d\}, \{a, b\}, \{c, d\}, \{b\}, X\}$. Define $F: (X, \tau) \to (X, \sigma)$ by F(a) ={ a, d}, F(b) ={a, c} , F(c) = {c, d} and F(d) = {a, b}. F is upper slightly p-continuous.

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