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Research Article

SOME GENERALIZATIONS OF ENESTRöM – KAKEYA THEOREM

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ABSTRACT
In this paper we establish some more generalizations of Eneström – Kakeya theorem a classical result in the theory of distribution of zeros of polynomials .Besides many consequences our results considerably improve the bounds in some cases as well. Mathematics subject classification (2000):30C10, 30C15.

Key Words: *Polynomial, Zeros, Eneström – Kakeya Theorem, Generalizations*

INTRODUCTION
The fallowing classical result known as Eneström – Kakeya theorem Marden (1966); Rahman and Scmeisser (2002) is famous in the theory of distribution of zeros of polynomials.
Theorem A:
If P (z) $=\sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$$

Then P (z) has all its zeros in the disk $|z|\leq 1$.
In the literature attempts have been made to extend and generalize the Eneström - Kakeya theorem. Joyal, Labelle and Rahman (1967) extended it to the polynomials with general monotonic coefficients by showing that, if the coefficients of the polynomial satisfy the condition

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \,,$$

Then all the zeros of P (z) are contained in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and zargar (1996) relaxed the hypothesis of Theorem A and proved:
Theorem B: If P (z) $=\sum_{r=0}^{n} a_r z^r$ *is* a polynomial of degee n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_2 \geq a_1 \geq a_0 \,.$$

Then P (z) has all its zeros in the disk.

$$|z+\lambda-1| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}.$$

On the other hand Govil and Rahman (1968) extended this theorem to the polynomials with complex coefficients, assuming that the moduli of coefficients are monotonic, and proved the following:
Theorem C: If P (z) $=\sum_{r=0}^{n} a_r z^r$ *is* a polynomial of degee n such that for some t > 0
$|a_n| \geq t|a_{n-1}| \geq t^2|a_{n-2}| \geq \cdots \geq t^{n-2}|a_2| \geq t^{n-1}|a_1| \geq t^n|a_0|$.
Then all the zeros of P (z) lie in the disk $|z| \leq \frac{k_1}{t}$, where k_1 is the greatest positive root of the equation

$$K^{N+1} - 2K^n + 1 = 0.$$

In the same paper they also proved that, if P (z) $=\sum_{r=0}^{n} a_r z^r$ *is* a polynomial of degee n such that for someβ.

$$\text{IArg } a_r - \beta\text{I} \leq \alpha \leq \frac{\pi}{2}; \qquad r=0, 1, 2\ldots n$$

and

$$|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0| \,,$$

Then all the zeros of P (z) lie in the disk

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{i=0}^{n} |a_i|,$$

Recently as further generalizations of the Eneström – Kakeya theorem Choo (2011) proved:

Theorem D: let P (z) $=\sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n such that for some $\lambda \neq 1$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$,

$$a_n \geq a_{n-1} \geq \ldots \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq \ldots \geq a_0 .$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P (z) lie in the disk $|z| \leq k_1$, where k_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

Where

$$\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n} \quad \text{and} \quad \delta_1 = \frac{a_n+(\lambda-1)a_{n-k}-a_0+|a_0|}{|a_n|}.$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of P (z) lie in the disk $|z| \leq k_2$, where k_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0,$$

Where

$$\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n} \quad \text{and} \quad \delta_2 = \frac{a_n+(1-\lambda)a_{n-k}-a_0+|a_0|}{|a_n|}.$$

Motivated by his method of proof, we in this paper prove the fallowing results which include Theorem D and some other generalizations of Eneström - Kakeya theorem as special cases.

Theorem 1: Let P (z) $=\sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n such that for some real μ and k, $1 \leq k \leq n$.

$$a_n \geq a_{n-1} \geq \ldots \geq a_{n-k+1} \geq \mu + a_{n-k} \geq a_{n-k-1} \geq \ldots \geq a_0 > 0.$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P (z) lie in the disk $|z| \leq k_1$, where k_1 is the greatest positive root of the equation

$$R^{k+1} - \left(1+\frac{\mu}{a_n}\right)R^k - \frac{|\mu|}{a_n} = 0.$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of P (z) lie in the disk $|z| \leq k_2$, where k_2 is the greatest positive root of the equation

$$R^k - \left(1+\frac{\mu}{a_n}\right)R^{k-1} - \frac{|\mu|}{a_n} = 0.$$

Remark 1: If we take $\mu = (\lambda-1)\, a_{n-k}$, such that $\lambda \neq 1$, in Theorem 1, we get Theorem D. Also for μ=0 this theorem reduces to Eneström - Kakeya theorem. Further If we take k=0 in Theorem 1 and there by assumes

$$\mu + a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0,$$

Then we get the following:

Corollary 1: If P (z) is a polynomial of degree n such that for some $\mu \geq 0$

$$\mu + a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0,$$

Then all the zeros of P (z) lie in

$$|z| \leq \frac{2\mu}{a_n} + 1$$

Remark 2: If we take μ= 0 in Corollary 1, we get Eneström - Kakeya theorem. Whereas if we replace μ by $a_n(\lambda-1)$ in Corollary 1 we get the following:

Corollary 2.If P (z) $=\sum_{r=0}^{n} a_r z^r$ *is* a polynomial of degee n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \ldots \geq a_2 \geq a_1 \geq a_0 > 0,$$

Then all the zeros of P (z) lie in

$$|z| \leq 2\lambda - 1$$

We next prove

Theorem 2: Let P(z) $= \sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n with $Re(a_r) = \alpha_r$ and $Im(a_r) = \beta_r$ r = 0,1,2,…,n and assume that for some μ and k , $1 \leq k \leq n$,

$$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_{n-k+1} \geq \mu + \alpha_{n-k} \geq \ldots \geq \alpha_1 \geq \alpha_0 ,$$

$$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 .$$

If $\alpha_{n-k-1} > \alpha_{n-k}$,then all the zeros of P (z) lie in the disk $|z| \leq k_1$, where k_1 is the greatest positive root of the equation

$$R^{k+1}-\left(\frac{\alpha_n+\beta_n+\mu}{|a_n|}\right)R^k-\frac{|\mu|}{|a_n|}=0$$

If $\alpha_{n-k}>\alpha_{n-k+1}$, then all the zeros of P (z) lie in the disk $|z|\leq k_2$, where k_2 is the greatest positive root of the equation

$$R^k-\frac{\alpha_n+\beta_n+\mu}{|a_n|}R^{k-1}-\frac{|\mu|}{|a_n|}=0$$

We also prove:

Theorem 3: Let $P(z)=\sum_{r=0}^{n}a_r z^r$ be a polynomial of degree n such that for some real β

$$|\text{Arg } a_r-\beta|\leq\alpha\leq\frac{\pi}{2};\qquad r=0,1\ldots n$$

And for some real $\mu>0$.

$|a_n|\geq|a_{n-1}|\geq\ldots\geq|a_{n-k+1}|\geq\mu+|a_{n-k}|\geq|a_{n-k-1}|\geq\ldots\geq|a_0|$.

If $|a_{n-k}|<|a_{n-k-1}|$, then all the zeros of P (z) lie in the disk $|z|\leq k1$. Here k_1 is the greatest positive root of the equation

$$R^{k+1}-\delta R^k-\frac{|\mu|}{|a_n|}=0,$$

Where

$$\delta=\frac{(|a_n|+|\mu+a_{n-k}|)(\cos\alpha+\sin\alpha)-(|a_{n-k}|+a_0|)\cos\alpha-(|a_{n-k}-|a_0|)\sin\alpha+2\sin\alpha\sum_{i=0}^{n-1}|a_i|+|a_0|}{|a_n|}.$$

If $|a_{n-k}|>|a_{n-k+1}|$ then all the zeros of P(z) lie in the disk $|z|\leq k_2$, where k_2 is the greatest positive root of the equation

$$R^k-\delta_1 R^{k-1}-\left|\frac{\mu}{a_n}\right|=0,$$

And

$$\delta_1=\frac{(|a_n|-|a_{n-k}-\mu|+|a_{n-k|}-a_0)\cos\alpha-\{(|a_{n-k}|+a_0|-(|a_{n-k}-\mu|+|a_n|+2\sum_{i=0}^{n-1}|a_i|\}\sin\alpha+|a_{0|}}{|a_n|}.$$

Lemma

We need the following lemma which is due to Govil and Rahman (1968).

Lemma 1: Consider two complex numbers b_0 and b_1, such that $|b0|\geq|b_1|$, if $|\text{Arg } b_r-\beta|\leq\alpha\leq\frac{\pi}{2}$; r=0, 1 then

$$|b_0-b_1|\leq(|b_0|-|b_1|)\cos\alpha+(|b_0|+|b_1|)\sin\alpha.$$

Proofs of Theorems

Proof of Theorem 1: For μ=0, the result reduces to Eneström- Kakeya theorem, therefore we suppose that $\mu\neq 0$. Consider a polynomial

$f(z)=(1-z)P(z)$

$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}+\ldots+(a_1-a_0)z+a_0$.

If $a_{n-k-1}>a_{n-k}$, then $a_{n-k+1}>a_{n-k}$ and f (z) can be written as

$f(z)=-a_n z^{n+1}-\mu z^{n-k}+(a_n-a_{n-1})z^n+\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(\mu+a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots$
$+(a_1-a_0)z+a_0$.

Therefore for $|z|>1$, we have

$|f(z)|\geq|a_n z^{n+1}+\mu z^{n-k}|-|z|^n|(a_n-a_{n-1})+\ldots+\left(\frac{a_{n-k+1}-a_{n-k}}{z^{k-1}}\right)+\left(\frac{\mu+a_{n-k}-a_{n-k-1}}{z^k}\right)+\ldots+\left(\frac{a_1-a_0}{z^{n-1}}\right)+\frac{a_0}{z^n}|$

$\geq|z|^{n-k}|a_n z^{k+1}+\mu|-|z|^n\{|a_n-a_{n-1}|+\ldots+\left|\frac{a_{n-k+1}-a_{n-k}}{z^{k-1}}\right|+\left|\frac{\mu+a_{n-k}-a_{n-k-1}}{z^k}\right|+\ldots+\left|\frac{a_1-a_0}{z^{n-1}}\right|+\left|\frac{a_0}{z^n}\right|\}$

$>|z|^{n-k}|a_n z^{k+1}+\mu|-|z|^n\{a_n-a_{n-1}+a_{n-1}-a_{n-2}+\ldots+a_{n-k+1}-a_{n-k}+\mu+a_{n-k}-a_{n-k-1}+\ldots+a_1-a_0+a_0\}$

$=|z|^{n-k}|a_n z^{k+1}+\mu|-|z|^n\{a_n+\mu\}$

>0,

If $|z^{k+1}+\frac{\mu}{a_n}|>(1+\frac{\mu}{a_n})|z|^k$. This inequality holds if

$$|z|^{k+1}-\frac{|\mu|}{a_n}>(1+\frac{\mu}{a_n})|z|^k.$$

Therefore in this case all the zeros of f (z) with modulus greater than one lie in the disk $|z|\leq k_1$, where k_1 is the greatest positive root of the equation

$$f_1(R) = R^{k+1} - (1 + \frac{\mu}{a_n})R^k - \frac{|\mu|}{a_n} = 0$$

Since $a_{n-k-1} > a_{n-k}$, therefore $\mu > 0$ and in this case $f_1(1) = -\frac{2\mu}{a_n} < 0$. Hence $k_1>1$.This shows that those zeros of f (z) whose modulus is less than or equal to one are already contained in the disk $|z| \le k_1$. Hence all the zeros of f (z) lie in $|z| \le k_1$, where k_1 is the greatest positive root of the equation

$$f_1(R) = R^{k+1} - (1 + \frac{\mu}{a_n})R^k - \frac{|\mu|}{a_n} = 0$$

Again, if $a_{n-k} > a_{n-k+1}$, then $a_{n-k} > a_{n-k-1}$ and f(z) can be written as

$$f(z) = -a_nz^{n+1} - \mu z^{n-k+1} + (a_n-a_{n-1}) z^n+\ldots+(a_{n-k+1}-a_{n-k}+\mu) z^{n-k+1}+(a_{n-k} - a_{n-k-1})z^{n-k}+\ldots+(a_1-a_0)z+a_0.$$

Now, for $|z| >1$,

$$|f(z)| \ge |z|^{n-k+1} |a_nz^k+\mu| - |z|^n|(a_n-a_{n-1})+ \ldots + (\frac{a_{n-k+1}-a_{n-k}+\mu}{z^{k-1}})+(\frac{a_{n-k}-a_{n-k-1}}{z^k})+ \ldots +(\frac{a_1-a_0}{z^{n-1}})+ \frac{a_0}{z^n}|.$$

$$\ge |z|^{n-k+1} |a_nz^k+\mu| - |z|^n\{| a_n-a_{n-1}| + \ldots + \frac{|a_{n-k+1}-a_{n-k}+\mu|}{|z|^{k-1}} + \frac{|a_{n-k}-a_{n-k-1}|}{|z|^k} + \ldots + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}.$$

$$> |z|^{n-k+1} |a_nz^k+\mu| - |z|^n\{a_n-a_{n-1}+ \ldots +a_{n-k+1} - a_{n-k} + \mu + a_{n-k} - a_{n-k-1}+ \ldots + -a_1 - a_0 + a_0\}.$$

$$= |z|^{n-k+1} |a_nz^k+\mu| - |z|^n\{ a_n+\mu\}$$

$$\geq 0.$$

If

$$|z^k + \frac{\mu}{a_n}| > (1 + \frac{\mu}{a_n}) |z|^{k-1}$$

This inequality holds if

$$|z|^k - \frac{|\mu|}{a_n} > (1+\frac{\mu}{a_n}) |z|^{k-1}$$

Hence all the zeros of f(z) with modulus greater than one lie in the disk $|z| \le k_2$, where k_2 is the greatest positive root of the equation

$$R^k - (1 + \frac{\mu}{a_n}) R^{k-1} - \frac{|\mu|}{a_n} = 0.$$

Now, as in the first case, it can be easily shown that $k_2 \ge 1$. Therefore, the zeros of f(z) with modulus less than or equal to one are already contained in the disk $|z| \le k_2$. Finally we note that every zero of P(z) is also a zero of f(z) and the proof of Theorem 1 is complete.

Proof of Theorem 2: Consider the polynomial

$$f(z) = (1-z)P(z)$$
$$= -a_nz^{n+1}+ (a_n-a_{n-1})z^n+\ldots+ (a_{n-k+1}-a_{n-k}) z^{n-k+1}+ (a_{n-k} - a_{n-k-1})z^{n-k} +\ldots+(a_1-a_0)z+a_0.$$
$$= -a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+\ldots+(\alpha_{n-k+1}-\alpha_{n-k}) z^{n-k+1}+(\alpha_{n-k} - \alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2}) z^{n-k-1}+\ldots+(\alpha_1-\alpha_0)z+\alpha_0$$
$$+ i\{(\beta_n-\beta_{n-1})z^n+\ldots+(\beta_1-\beta_0)z+\beta_0\}.$$

Now if $\alpha_{n-k-1} > \alpha_{n-k}$ then $\alpha_{n-k+1} > \alpha_{n-k}$ and f(z) can be written as

$$f(z) = -a_nz^{n+1}-\mu z^{n-k} + (\alpha_n-\alpha_{n-1})z^n+\ldots+(\alpha_{n-k+1}- \alpha_{n-k}) z^{n-k+1}+(\mu+\alpha_{n-k} - \alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2}) z^{n-k-1}+\ldots$$
$$+(\alpha_1-\alpha_0)z+\alpha_0 +i\{(\beta_n-\beta_{n-1})z^n+\ldots+(\beta_1-\beta_0)z+ \beta_0\}.$$

Therefore for $|z| >1$, we have

$$|f(z)|\ge |a_nz^{n+1}+\mu z^{n-k}| - |z|^n|(\alpha_n-\alpha_{n-1}) + \ldots+(\frac{\alpha_{n-k+1}-\alpha_{n-k}}{z^{k-1}})+(\frac{\mu+\alpha_{n-k}-\alpha_{n-k-1}}{z^k})+ \ldots+(\frac{\alpha_1-\alpha_0}{z^{n-1}})$$
$$+ \frac{\alpha_0}{z^n} + i\{(\beta_n-\beta_{n-1})+\ldots+\frac{(\beta_1-\beta_0)}{z^{n-1}}+\frac{\beta_0}{z^n}\}|$$
$$\ge |z|^{n-k} |a_nz^{k+1}+\mu|-|z|^n\{| \alpha_n-\alpha_{n-1} | +|\frac{\alpha_{n-1}-\alpha_{n-2}}{z}| + \ldots+ |\frac{\alpha_{n-k+1}-\alpha_{n-k}}{z^{k-1}}| + |\frac{\mu+\alpha_{n-k}-\alpha_{n-k-1}}{z^k}|+$$
$$\ldots + |\frac{\alpha_1-\alpha_0}{z^{n-1}}| + \frac{\alpha_0}{z^n}+|\beta_n-\beta_{n-1}|+\ldots+|\frac{\beta_1-\beta_0}{z^{n-1}}| + |\frac{\beta_0}{z^n}|\}$$
$$> |z|^{n-k} |a_nz^{k+1}+\mu|-|z|^n\{\alpha_n-\alpha_{n-1} +\alpha_{n-1} - \alpha_{n-2}+\ldots+ \alpha_{n-k+1}-\alpha_{n-k} +\mu+\alpha_{n-k} - \alpha_{n-k-1} + \ldots+\alpha_1- \alpha_0+ \alpha_0 + \beta_n- \beta_{n-1} +\ldots+\beta_1-\beta_0+ \beta_0\} = |z|^{n-k} | a_nz^{k+1}+\mu|-|z|^n\{\alpha_n+\beta_n +\mu\}$$
$$\geq 0,$$

If

$$|z^{k+1}+ \frac{\mu}{a_n}|> (\frac{\alpha_n+\beta_n+\mu}{|a_n|})|z|^k.$$

But *this inequality holds if*

$$|z|^{k+1} - \frac{\mu}{|a_n|} > \left(\frac{\mu+\alpha_n+\beta_n}{|a_n|}\right)|z|^k.$$

This shows that all the zeros of f(z) with modulus greater than one lie in the disk $|z| \le k_1$, where k_1 is the greatest positive root of the equation

$$R^{k+1} - \left(\frac{\mu+\alpha_n+\beta_n}{|a_n|}\right)R^k - \frac{\mu}{|a_n|} = 0.$$

As in the case of Theorem 1, It can be shown that $k_1>1$. Hence all the zeros of f(z) with modulus less than or equal to one already lie in the disk $|z| \le 1$ and the proof of the first part is complete.
Now suppose that $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$, and f(z) can be written as

$f(z) = -a_n z^{n+1} - \mu z^{n-k+1} + (\alpha_n-\alpha_{n-1})z^n + \ldots + (\alpha_{n-k+1}-\alpha_{n-k}+\mu)z^{n-k+1} + (\alpha_{n-k}-\alpha_{n-k-1})z^{n-k} + \ldots + (\alpha_1-\alpha_0)z + \alpha_0 +$
$\quad i\{(\beta_n-\beta_{n-1})z^n + \ldots + (\beta_1-\beta_0)z + \beta_0$

If $|z|>1$, then

$|f(z)| \ge |a_n z^{n+1} + \mu z^{n-k+1}| - |z|^n\{|\alpha_n-\alpha_{n-1}| + \ldots + \left|\frac{\alpha_{n-k+1}-\alpha_{n-k}+\mu}{z^{k-1}}\right| + \left|\frac{\alpha_{n-k}-\alpha_{n-k-1}}{z^k}\right| + \ldots +$
$\quad \left|\frac{\alpha_1-\alpha_0}{z^{n-1}}\right| + \left|\frac{\alpha_0}{z^n}\right| + |\beta_n-\beta_{n-1}| + \ldots + \left|\frac{\beta_{n-k}-\beta_{n-k-1}}{z^k}\right| + \ldots + \left|\frac{\beta_1-\beta_0}{z^{n-1}}\right| + \left|\frac{\beta_0}{z^n}\right|\}$
$\quad = |z|^{n-k+1}\,|a_n z^k + \mu| - |z|^n(\alpha_n+\beta_n+\mu)$

$> \quad 0$

If $\quad |z^k + \frac{\mu}{a_n}| > |z|^{k-1}\frac{\alpha_n+\beta_n+\mu}{|a_n|}.$

But this inequality holds if

$$|z|^k - \frac{\mu}{|a_n|} > \frac{\alpha_n+\beta_n+\mu}{|a_n|}|z|^{k-1}.$$

Hence all the zeros of f(z)with modulus greater than one lie in the disk $|z| \le K_2$, where K_2 is the greatest root of the equation

$$R^k - \frac{\alpha_n+\beta_n+\mu}{|a_n|}R^{k-1} - \frac{\mu}{|a_n|} = 0.$$

Again it can be easily verified that $k_2>1$ and therefore all the zeros of f(z) with modulus less than or equal to one are already contained in the disk $|z| \le k_2$.Finally we note that every zero of P(z) is also a zero of f(z), the proof of the Theorem 2 is complete.

Proof of Theorem 3: Consider the polynomial

$f(z) = (1-z)P(z)$
$\quad = -a_n z^{n+1} + (a_n-a_{n-1})z^n + \ldots + (a_{n-k+1}-a_{n-k})z^{n-k+1} + (a_{n-k}-a_{n-k-1})z^{n-k} + (a_{n-k-1}-a_{n-k-2})z^{n-k-1} + \ldots + (a_1-a_0)z + a_0.$

If $|a_{n-k-1}| > |a_{n-k}|$then $|a_{n-k+1}| > |a_{n-k}|$and f(z) can be written as

$f(z) = -a_n z^{n+1} - \mu z^{n-k} + (a_n-a_{n-1})z^n + \ldots + (a_{n-k+1}-a_{n-k})z^{n-k-1} + (\mu+a_{n-k}-a_{n-k-1})z^{n-k} + (a_{n-k-1}-a_{n-k-2})z^{n-k-1} + \ldots +$
$\quad (a_1-a_0)z + a_0$

For $|z|>1$

$|f(z)| \ge |a_n z^{n+1} + \mu z^{n-k}| - |z|^n|(a_n-a_{n-1}) + \ldots + \left(\frac{a_{n-k+1}-a_{n-k}}{z^{k-1}}\right) + \frac{(\mu+a_{n-k}-a_{n-k})}{z^k} + \ldots + \left(\frac{a_1-a_0}{z^{n-1}}\right) + \frac{a_0}{z^n}|$
$\quad \ge |z|^{n-k}\,|a_n z^{k+1} + \mu| - |z|^n\{|a_n-a_{n-1}| + \ldots + \left|\frac{a_{n-k+1}-a_{n-k}}{z^{k-1}}\right| + \left|\frac{\mu+a_{n-k}-a_{n-k}}{z^k}\right| + \ldots + \left|\frac{a_1-a_0}{z^{n-1}}\right| +$
$\quad \left|\frac{a_0}{z^n}\right|\}$
$\quad > |z|^{n-k}\,|a_n z^{k+1} + \mu| - |z|^n\{|a_n-a_{n-1}| + \ldots + |a_{n-k+1}-a_{n-k}| + |\mu+a_{n-k}-a_{n-k-1}| + \ldots + |a_1-a_0| + |a_0|$

Using lemma 1, this gives

$|f(z)| \ge |z|^{n-k}\,|a_n z^{k+1} + \mu| - |z|^n\{(|a_n|-|a_{n-1}|)\cos\alpha + (|a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha +$
$\quad (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots + (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + |a_{n-k}|)\sin\alpha + (|\mu + a_{n-k}| - |a_{n-k-1}|)\cos\alpha +$
$\quad (|\mu + a_{n-k}| + |a_{n-k-1}|)\sin\alpha + \ldots + (|a_1| - |a_0|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha\} + |a_0|$
$\quad = |z|^{n-k}\,|a_n z^{k+1} + \mu| - |z|^n\{(|a_n| - |a_{n-k}| + |\mu + a_{n-k}| - |a_0|)\cos\alpha + (|a_n| + 2\sum_{r=o}^{n}|a_r| - |a_{n-k}| +$
$\quad |\mu + a_{n-k}| + |a_0|)\sin\alpha + |a_0|$
$\quad \ge |z|^{n-k}\,|a_n z^{k+1} + \mu| - |z|^n\{(|a_n| + |\mu + a_{n-k}|)(\cos\alpha + \sin\alpha) - (|a_{n-k}| + |a_0|)\cos\alpha - (|a_{n-k}| - |a_0|)\sin\alpha +$
$\quad 2\sin\alpha\sum_{r=o}^{n}|a_r| + |a_0|\}$

$> \quad 0,$

If

$$|z^{k+1}+\frac{\mu}{a_n}|>|z|^k\delta ,$$

Where

$$\delta=\frac{(|a_n|+|\mu+a_{n-k}|)(\cos\alpha+\sin\alpha)-(|a_{n-k}|+|a_0|)\cos\alpha-(|a_{n-k}-|a_0|)\sin\alpha+2\sin\alpha\sum_{i=0}^{n-1}|a_i|+a_0}{|a_n|}$$

This inequality holds if

$$|z|^{k+1}-|\frac{\mu}{a_n}|>|z|^k\delta$$

This shows that all the zeros of f(z)with modulus greater than one lie in the disk |z|< k_1, where k_1 is the greatest positive root of the equation

$$R^{k+1}-\delta R^k-|\frac{\mu}{a_n}|=0$$

Also, it is easily seen that $k_1 \geq 1$,and therefore all the zeros of f(z) with modulus less than or equal to one are already in the disk $|z|\leq k_1$

Now consider the case $|a_{n-k}|>|a_{n-k+1}|$, then $|a_{n-k}|>|a_{n-k-1}|$ we have

$f(z) = -a_nz^{n+1}-\mu z^{n-k+1}+(a_n-a_{n-1})z^n+\ldots+(a_{n-k+1}-a_{n-k}+\mu)z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots+(a_1-a_0)z+a_0$

For IzI >1

$$|f(z)|\geq|a_nz^{n+1}+\mu z^{n-k+1}|-|z|^n|(a_n-a_{n-1})+--+(\frac{a_{n-k+1}-a_{n-k}+\mu}{z^{k-1}})+(\frac{a_{n-k}-a_{n-k+1}}{z^k})+---+(\frac{a_1-a_0}{z^{n-1}})+\frac{a_0}{z^n}|$$

$$\geq|a_nz^{n+1}+\mu z^{n-k+1}|-|z|^n\{|a_n-a_{n-1}|+--+|\frac{a_{n-k+1}-a_{n-k}+\mu}{z^{k-1}}|+|\frac{a_{n-k}-a_{n-k+1}}{z^k}|+---+|\frac{a_1-a_0}{z^{n-1}}|+|\frac{a_0}{z^n}|\}$$

$$\geq|z|^{n-k+1}|a_nz^k+\mu|-|z|^n\{|a_n-a_{n-1}|+---+|a_{n-k+1}-a_{n-k}+\mu|+|a_{n-k}-a_{n-k-1}|+---+|a_1-a_0|+|a_0|\}$$

This gives by lemma 1,

$|f(z)|\geq|z|^{n-k+1}|a_nz^k+\mu|-|z|^n\{(|a_n|-|a_{n-1}|)\cos\alpha+|a_n|+|a_{n-1})\sin\alpha+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+\ldots+(|a_{n-k+1}|-|a_{n-k}-\mu)\cos\alpha+(|a_{n-k+1}|+|a_{n-k}-\mu|)\sin\alpha+(|a_{n-k}|-|a_{n-k-1}|)\cos\alpha+(|a_{n-k}|+|a_{n-k-1}|)\sin\alpha+\ldots+(|a_1|-|a_0|)\cos\alpha+(|a_1|+|a_0|)\sin\alpha\}+|a_0|$

$$=|z|^{n-k+1}|a_nz^k+\mu|-|z|^n\delta_1$$

➢ 0 ,

If

$$|z^k+\frac{\mu}{a_n}|>\delta_1|z|^{k-1}$$

Where

$$\delta_1=\frac{(|a_n|-|a_{n-k}-\mu|+|a_{n-k|}-a_0)\cos\alpha-\{(|a_{n-k}|+a_0-|(|a_{n-k}-\mu|+|a_n|+2\sum_{i=0}^{n-1}|a_i|\}\sin\alpha+a_0}{a_n}$$

This above inequality holds if

$$|z|^k-|\frac{\mu}{a_n}|>\delta_1|z|^{k-1}$$

This shows that all the zeros of f(z) whose modulus greater than 1 lie in the disk $|z|\leq k_2$ where $k_2$is the greatest positive root of the equation,

$$R^k-\delta_1R^{k-1}-|\frac{\mu}{a_n}|=0$$

Again it can be shown that k_2>1and therefore those zeros of f(z) whose modulus is less than or equalto 1also lie in the disk $|z|\leq k_2$.Finally we note that every zero of P(z) is also a zero of f(z), and the Theorem 3 is proved completely.

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