*International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)*
*An Online International Journal Available at http://www.cibtech.org/jpms.htm*
*2012 Vol. 2 (2) April- June, pp.169-176/ Natarajan*
***Research Article***

# SOME PROPERTIES OF THE (M, $\lambda_n$) METHOD OF SUMMABILITY IN ULTRAMETRIC FIELDS

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**ABSTRACT**
In the present paper, K denotes a complete, non-trivially valued, ultrametric field. Infinite matrices, sequences and series have entries in K. The purpose of this paper is to prove some interesting properties of the (M, $\lambda_n$) method (or the M-method), introduced earlier by Natarajan (2002), in such a field K.

**2000 Mathematics Subject Classification:** 40, 46.

***Key Words:*** *Regular summability method, the (M, $\lambda_n$) method (or the M-method), consistency, limitation theorem, inclusion theorem, equivalence, Tauberian theorem*

## INTRODUCTION

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric field. Infinite matrices, sequences and series have entries in K.

Given an infinite matrix A = $(a_{nk})$, n, k = 0, 1, 2, ... and a sequence x = $\{x_k\}$, k = 0, 1, 2, ..., by the A-transform of the sequence x = $\{x_k\}$, we mean the sequence A(x) = $\{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,....,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that x = $\{x_k\}$ is summable A or A-summable to $\ell$. If $\lim_{n\to\infty} (Ax)_n = \ell$, whenever $\lim_{k\to\infty} x_k = \ell$, we say that the matrix method A is regular. The following theorem, which gives necessary and sufficient conditions for A to be regular in terms of its entries, is well-known (see, for instance, Monna (1963)).

**Theorem 1.1.** A = $(a_{nk})$ is regular if and only if

(i) $\sup_{n,k} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = 0, \quad k = 0,1,2,...;$

and

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1.$

An infinite series $\sum_{k=0}^{\infty} x_k$ is said to be A-summable to $\ell$ if $\{s_n\}$ is A-summable to $\ell$ where $s_n = \sum_{k=0}^{n} x_k$, n = 0, 1, 2, ... .

The M-method of summability was introduced earlier by Natarajan and some of its properties were studied in Natarajan (2002). Since the M-method depends on a sequence $\{\lambda_n\}$ in K, we shall henceforth call the M-method as (M, $\lambda_n$) method. In the present paper, we shall study some interesting properties of the (M, $\lambda_n$) method.

**Definition 1.1.** Let $\{\lambda_n\}$ be a sequence in K such that $\lim_{n\to\infty} \lambda_n = 0$. The (M, $\lambda_n$) method is defined by the infinite matrix A = $(a_{nk})$, where

$$a_{nk} = \begin{cases} \lambda_{n-k}, & k \le n; \\ 0, & k > n. \end{cases}$$

**Remark 1.1.** In this context, we note that the (M, $\lambda_n$) method reduces to the Y-method of Srinivasan (see Srinivasan (1965)), when K = $\mathbb{Q}_p$, $\lambda_0 = \lambda_1 = \frac{1}{2}$ and $\lambda_n = 0$, $n \ge 2$.

## REGULARITY, CONSISTENCY AND LIMITATION THEOREMS

**Theorem 2.1.** The method (M, $\lambda_n$) is regular if and only if $\sum_{n=0}^{\infty} \lambda_n = 1$.

*Proof*. Let (M, $\lambda_n$) be regular. Since $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n$ converges.

Now, $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$. i.e., $\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{n-k} = 1$, i.e., $\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_k = 1$, i.e., $\sum_{k=0}^{\infty} \lambda_k = 1$. Conversely, if $\sum_{n=0}^{\infty} \lambda_n = 1$, $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} \sum_{k=0}^{n} \lambda_k = \sum_{k=0}^{\infty} \lambda_k = 1$.

Since $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} \lambda_{n-k} = 0$, k = 0, 1, 2, ... .

Since $\{\lambda_n\}$ is bounded, it follows that $\sup_{n,k} |a_{nk}| < \infty$. In view of Theorem 1.1, (M, $\lambda_n$) = $(a_{nk})$ is regular. □

**Definition 2.1.** Two matrix methods A = $(a_{nk})$, B = $(b_{nk})$ are said to be consistent, if whenever x = $\{x_k\}$ is A-summable to s and B-summable to t, then s = t.

**Theorem 2.2.** Any two regular methods (M, $\lambda_n$), (M, $\mu_n$) are consistent.

*Proof*. Let $(M, \lambda_n)$, $(M, \mu_n)$ be two regular methods. Then $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mu_n = 0$. Let $\gamma_n = \lambda_0 \mu_n + \lambda_1 \mu_{n-1} + \cdots + \lambda_n \mu_0$, $n = 0, 1, 2, \ldots$ . Let $u_n = \lambda_n x_0 + \lambda_{n-1} x_1 + \cdots + \lambda_0 x_n \to s$, $n \to \infty$ and $v_n = \mu_n x_0 + \mu_{n-1} x_1 + \cdots + \mu_0 x_n \to t$, $n \to \infty$.

Now,

$$\begin{aligned}
w_n &= \gamma_0 x_n + \gamma_1 x_{n-1} + \cdots + \gamma_n x_0 \\
&= (\lambda_0 \mu_0) x_n + (\lambda_0 \mu_1 + \lambda_1 \mu_0)x_{n-1} + \cdots + (\lambda_0 \mu_n + \lambda_1 \mu_{n-1} + \cdots + \lambda_n \mu_0) x_0 \\
&= \lambda_0 [ \mu_0 x_n + \mu_1 x_{n-1} + \cdots + \mu_n x_0] + \lambda_1 [ \mu_0 x_{n-1} + \mu_1 x_{n-2} + \cdots + \mu_{n-1} x_0] \\
&\qquad + \cdots + \lambda_n [ \mu_0 x_0] \\
&= \lambda_0 v_n + \lambda_1 v_{n-1} + \cdots + \lambda_n v_0 \\
&= [\lambda_0 (v_n-t) + \lambda_1 (v_{n-1}-t) + \cdots + \lambda_n (v_0-t)] + t ( \lambda_0 + \lambda_1 + \cdots + \lambda_n).
\end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n = 0 = \lim_{n\to\infty} (v_n - t)$,

$$\lim_{n\to\infty} \left[\lambda_0(v_n - t) + \lambda_1(v_{n-1} - t) + \cdots + \lambda_n(v_0 - t)\right] = 0,$$

in view of Theorem 1 of Natarajan (1978). Thus

$$\lim_{n\to\infty} w_n = t\sum_{n=0}^{\infty}\lambda_n = t,$$

since $\sum_{n=0}^{\infty}\lambda_n = 1$, $(M, \lambda_n)$ being regular, using Theorem 2.1. In a similar fashion, we can prove that $\lim_{n\to\infty} w_n = s$, so that $s = t$. Consequently $(M, \lambda_n)$, $(M, \mu_n)$ are consistent. □

**Theorem 2.3.** (Limitation theorem) If $\{x_n\}$ is $(M, \lambda_n)$ summable, where $|\lambda_n| < |\lambda_0|$, $n \geq 1$, then $\{x_n\}$ is bounded.

*Proof*. Note that, since $|\lambda_0| > |\lambda_1|$, $\lambda_0 \neq 0$. Let

$$u_n = \lambda_0 x_n + \lambda_1 x_{n-1} + \cdots + \lambda_n x_0, \quad n = 0, 1, 2, \ldots .$$

Then $\{u_n\}$ converges and so bounded.

Thus $|u_n| \leq M$, $n = 0, 1, 2, \ldots$ .

$$u_0 = \lambda_0 x_0$$

so that

$$|x_0| \leq \left|\frac{u_0}{\lambda_0}\right| \leq \frac{M}{|\lambda_0|}.$$

Let now

$$|x_k| \leq \frac{M}{|\lambda_0|}, \quad k = 0,1,2,\ldots,(n-1).$$

$$\lambda_0 x_n = \lambda_1 x_{n-1} + \lambda_2 x_{n-2} + \cdots + \lambda_n x_0 - u_n ,$$

$$|\lambda_0 x_n| \leq \max\left\{|\lambda_0| \frac{M}{|\lambda_0|}, M\right\} = M,$$

from which it follows that $|x_n| \leq \frac{M}{|\lambda_0|}$. This completes the induction. Thus $\{x_n\}$ is bounded.

□

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Consequently we have the following result.

**Theorem 2.4.** A power series $\sum_{n=0}^{\infty} a_n x^n$ is not (M, $\lambda_n$) summable outside its circle of convergence, where $|\lambda_n| < |\lambda_0|$, $n \geq 1$.

*Proof*. Let $\sum_{n=0}^{\infty} a_n x^n$ be (M, $\lambda_n$) summable at $x = x_0$. Then using Theorem 2.3, there exists $M > 0$ such that

$$|a_n||x_0|^n \leq M,$$

$$\text{i.e.,} \quad |a_n| \leq \frac{M}{|x_0|^n},$$

$$\text{i.e.,} \quad |a_n|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|x_0|}.$$

So

$$\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} \leq \frac{1}{|x_0|}, \quad \text{since} \quad \lim_{n\to\infty} M^{\frac{1}{n}} = 1,$$

$$\text{i.e.,} \quad \frac{1}{\rho} \leq \frac{1}{|x_0|},$$

$$\text{i.e.,} \quad |x_0| \leq \rho,$$

$\rho$ being the radius of convergence, proving the theorem. □

**INCLUSION THEOREM AND EQUIVALENCE**

K(x) denotes the ultrametric algebra of all formal power series $\sum_{n=0}^{\infty} \alpha_n x^n$ for which $\lim_{n\to\infty} \alpha_n = 0$ under the norm

$$\left\| \sum_{n=0}^{\infty} \alpha_n x^n \right\| = \sup_{n\geq 0} |\alpha_n|.$$

The following lemma was proved in Van Rooij (1978) (p. 233, Corollary 6.39).

**Lemma 3.1.** An element $f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ of K(x) is invertible if and only if $|\alpha_n| < |\alpha_0|$, $n = 1, 2, \ldots$ .

**Theorem 3.1.** (Inclusion theorem) Let (M, $\lambda_n$), (M, $\mu_n$) be given methods where $|\lambda_n| < |\lambda_0|$, $n = 1, 2, \ldots$ . Then (M, $\lambda_n$) $\subseteq$ (M, $\mu_n$), i.e., whenever $\{s_n\}$ is summable

(M, $\lambda_n$) to s, then it is also summable (M, $\mu_n$) to s, if and only if $\sum_{n=0}^{\infty} k_n = 1$ where $\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{n=0}^{\infty} k_n x^n$.

*Proof*. Let $\lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n$, $\mu(x) = \sum_{n=0}^{\infty} \mu_n x^n$. The series on the right converge for $|x| < 1$, since $\lim_{n\to\infty} \lambda_n = 0 = \lim_{n\to\infty} \mu_n$. Since $|\lambda_n| < |\lambda_0|$, n = 1, 2, ..., $\lambda(x)$ is invertible in view of Lemma 3.1. Let $\{u_n\}$, $\{v_n\}$ be the (M, $\lambda_n$), (M, $\mu_n$) transforms of $\{s_n\}$ respectively. If $|x| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} v_n x^n &= \sum_{n=0}^{\infty} (\mu_0 s_n + \mu_1 s_{n-1} + \cdots + \mu_n s_0) x^n \\ &= \left(\sum_{n=0}^{\infty} \mu_n x^n\right)\left(\sum_{n=0}^{\infty} s_n x^n\right) \\ &= \mu(x) s(x).\end{aligned}$$

Similarly

$$\sum_{n=0}^{\infty} u_n x^n = \lambda(x) s(x), \quad \text{if} \quad |x| < 1.$$

Now,

$$k(x)\,\lambda(x) = \mu(x),$$
$$k(x)\,\lambda(x)\,s(x) = \mu(x)\,s(x),$$
$$\text{i.e., } k(x)\left(\sum_{n=0}^{\infty} u_n x^n\right) = \sum_{n=0}^{\infty} v_n x^n.$$

Thus

$$\begin{aligned} v_n &= k_0 u_n + k_1 u_{n-1} + \cdots + k_n u_0 \\ &= \sum_{j=0}^{\infty} a_{nj} u_j, \end{aligned}$$

where

$$a_{nj} = \begin{cases} k_{n-j}, & j \le n; \\ 0, & j > n. \end{cases}$$

If (M, $\lambda_n$) $\subseteq$ (M, $\mu_n$), ($a_{nj}$) is regular. So $\lim_{n\to\infty} a_{n0} = 0$,

i.e., $\lim_{n\to\infty} k_n = 0$ and so $\sum_{n=0}^{\infty} k_n$ converges. Also $\lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj} = 1$,

i.e., $\lim_{n\to\infty} \sum_{j=0}^{n} k_{n-j} = 1$, i.e., $\lim_{n\to\infty} \sum_{j=0}^{n} k_j = 1$, i.e., $\sum_{n=0}^{\infty} k_n = 1$. Conversely, if $\sum_{n=0}^{\infty} k_n = 1$, then it follows that $(a_{nj})$ is regular and so $\lim_{j\to\infty} u_j = s$ implies $\lim_{n\to\infty} v_n = s$, i.e., $(M, \lambda_n) \subseteq (M, \mu_n)$, completing the proof of the theorem.

□

As a consequence of Theorem 3.1, we have

**Theorem 3.2.** Let $(M, \lambda_n)$, $(M, \mu_n)$, be given methods with $|\lambda_n| < |\lambda_0|$, $|\mu_n| < |\mu_0|$, $n \geq 1$. Then $(M, \lambda_n)$ and $(M, \mu_n)$ are equivalent, i.e., $(M, \lambda_n) \subseteq (M, \mu_n)$ and vice versa, if and only if $\sum_{n=0}^{\infty} k_n = \sum_{n=0}^{\infty} h_n = 1$ where $\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{n=0}^{\infty} k_n x^n$ and $\frac{\lambda(x)}{\mu(x)} = h(x) = \sum_{n=0}^{\infty} h_n x^n$.

## TAUBERIAN THEOREMS

We need the following result in the sequel.

**Theorem 4.1.** (see Natarajan (1997), Theorem 1) Let A be any regular matrix method and $\sum_{n=0}^{\infty} a_n$ be A-summable to s. Let $\lim_{n\to\infty} a_n = \ell$. If A({n}) diverges, then $\sum_{n=0}^{\infty} a_n$ converges to s. In other words, $\lim_{n\to\infty} a_n = \ell$ is a Tauberian condition provided A({n}) diverges.

As a consequence of Theorem 4.1, we now have

**Theorem 4.2.** If $\sum_{n=0}^{\infty} a_n$ is $(M, \lambda_n)$ summable to s, $(M, \lambda_n)$ being regular and if $\lim_{n\to\infty} a_n = \ell$, then $\sum_{n=0}^{\infty} a_n$ converges to s.

*Proof.* In view of Theorem 4.1, it suffices to prove that {n} is not summable by the regular $(M, \lambda_n)$ method.

Let

$$u_n = \lambda_0 . n + \lambda_1 . (n-1) + \cdots + \lambda_{n-1} . 1 + \lambda_n . 0.$$

Then

$$u_n - u_{n-1} = \{\lambda_0 . n + \lambda_1 . (n-1) + \cdots + \lambda_{n-1} . 1\} - \{\lambda_0 . (n-1) + \lambda_1 . (n-2) + \cdots + \lambda_{n-2} . 1\}$$
$$= (\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}).$$

So

$$\lim_{n\to\infty} (u_n - u_{n-1}) = \sum_{n=0}^{\infty} \lambda_n = 1,$$

$(M, \lambda_n)$ being regular. Thus $\{u_n\}$ is not cauchy and so diverges, i.e., {n} is not summable $(M, \lambda_n)$, completing the proof. □

We now introduce the following.

**Definition 4.1.** Let $s = \{s_0, s_1, s_2, ...\}$, $\bar{s} = \{0, s_0, s_1, ...\}$, $s^* = \{s_1, s_2, ...\}$. A summability method A is said to be translative if $\bar{s}$ and $s^*$ are summable to $\ell$ whenever s is summable to $\ell$.

**Theorem 4.3.** Every $(M, \lambda_n)$ method is translative.
*Proof.* Writing $A \equiv (M, \lambda_n)$,

$$\begin{aligned}(A\bar{s})_n &= \lambda_n.0 + \lambda_{n-1}.s_0 + \lambda_{n-2}.s_1 + \cdots + \lambda_0.s_{n-1}\\ &= \lambda_{n-1}.s_0 + \lambda_{n-2}.s_1 + \cdots + \lambda_0.s_{n-1}\\ &= u_{n-1},\end{aligned}$$

where

$$u_n = \lambda_n . s_0 + \lambda_{n-1} . s_1 + \cdots + \lambda_0 . s_n, \qquad n = 0, 1, 2, \ldots .$$

So, if $u_n \to \ell$, $n \to \infty$, then $(A\bar{s})_n \to \ell$, $n \to \infty$. Also

$$\begin{aligned}(As^*)_n &= \lambda_n . s_1 + \lambda_{n-1} . s_2 + \cdots + \lambda_0 . s_{n+1}\\ &= (\lambda_{n+1} . s_0 + \lambda_n . s_1 + \cdots + \lambda_0 . s_{n+1}) - \lambda_{n+1}s_0\\ &= u_{n+1} - \lambda_{n+1} . s_0\\ &\to \ell,\ n \to \infty, \quad \text{since } u_{n+1} \to \ell,\ n \to \infty \text{ and } \lambda_n \to 0,\ n \to \infty.\end{aligned}$$

Thus the method $(M, \lambda_n)$ is translative. □

Using Theorem 3 of Natarajan (1997), we now have the following result.

**Theorem 4.4.** If $\sum_{n=0}^{\infty} a_n$ is summable to s by a regular $(M, \lambda_n)$ method and $\lim_{n\to\infty}(a_{n+1} - a_n) = \ell$, then $\sum_{n=0}^{\infty} a_n$ converges to s.

Using Theorem 5 of Natarajan (1997), we have the following result for regular $(M, \lambda_n)$ method.

**Theorem 4.5.** If $\sum_{n=0}^{\infty} a_n$ is summable by a regular $(M, \lambda_n)$ method, then the following Tauberian conditions are equivalent:
(i) $a_n \to \ell$, $n \to \infty$;
(ii) $\Delta a_n \equiv a_{n+1} - a_n \to \ell'$, $n \to \infty$.
If, further, $a_n \neq 0$, $n = 0, 1, 2, ...$, each of
(iii) $\dfrac{a_{n+1}}{a_n} \to 1$, $n \to \infty$;
and
(iv) $\dfrac{a_{n+2} + a_n}{a_{n+1}} \to 2$, $n \to \infty$

is a weaker Tauberian condition for the summability of $\sum_{n=0}^{\infty} a_n$ by a regular (M, $\lambda_n$) method.

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