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Research Article

ENERGY EQUATIONS FOR *N – TUPPLE* PENDULA SYSTEM

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ABSTRACT
A multiple chain pendula system constrained to move in a plane has been studied further within the framework of a generalized coordinate system by using Lagrangian formalism. Equations of motions for many body pendula systems have been derived. These equations concur very well with known data for $m \leq 5$. We confirm that equations of motion for any values of n and l can be generated from our general equation which presents interesting characteristics. Solutions to n-tuple pendulum system could be deduced.

***Key Words**: Lagrangian, Equations Of Motion, N-Tuple Pendula System*

INTRODUCTION
The study of the equations of motion for an n-tuple Pendula system using the Lagrange formulations is widely explored. Prichani *et al.,* (2012) worked on systems of up to five inter-linked masses restricted to move vertically in a plane whose equations of motion motivated us in making further studies on energy equations. These relations are generally used for the dynamics of masses in the determination of the dynamical variables (Tang, 2005). The use of the energy dependent Lagrange approach is a simpler methodology even for complicated mechanical systems that would be strenuous by Newtonian formulations. The resultant equations developed are second order partial differential equations in nature where time is an explicit function that is central in obtaining the frequencies, normal modes and the periodic times of the system.
Giarratana (1945) studied a set of general equations of motion for classical dynamical systems of variable mass, where the variation of mass may be caused by either a continuous deformation and motion of the deforming surface or a motion of the material points of the system or both. Warren *et al.,* (1989) showed that the Lagrange's equations depend exclusively on the difference between the total kinetic and potential energies of the system they presented the equations without using variational calculus a tool we find useful in the current method of developing equations of motion for the n-tupple pendulum
Leaderich *et al.,* (1992) studied the dynamics of planar chains consisting of coupled rigid bodies at which they derived the equations in intrinsic coordinates of the chain. The variational techniques were used in showing the existence of periodic solutions in many settings; Chang (1990). They used some standard methods in the calculus of variations to describe the periodic motions of the system. Compound pendula systems have been studied extensively with the evolution of the angles being governed by the Euler-Langrage equations from which the motion as viewed from the inertial frame, are periodic or quasi-periodic; Leaderich (1992), Koslov (1985). Other systems of compound pendula that have been studied is somewhat similar to that of Laederich (1992) and our current work in that it is just a chain suspended by one end in a gravitational field in which the dimension of the problem is reduced by eliminating the reference to an inertia frame; Koslov (1980). Furuta *et al.,* (1993) worked on the control of the multiple pendulums of varying constraints. In their study, the swing up control of pendulums was discussed by considering reach-ability of an unstable nonlinear control.
An open problem recognized and not solved by classical analysis methods is the quest for explicit and concise conditions for synchronization as a function of the topological, algebraic and the spectral graph properties of the network; Dr fler (2011). However the synchronization of coupled oscillators in classic models introduced by Kuramoto (2003) has been widely studied with numerous results that appears to have rich applications to various scientific areas; Strogatz (2000), Acebron (2005). The vast literature is

now elegantly reviewed for applicability and could have some similarities, though not entirely, to our coupled system of masses; Prichani *et al.*, (2012).
In this study we have considered a freely moving system whose mass displacements are very small but could have different mass sizes, varying inter-link lengths with masses at different angular displacements at different times. A rather comprehensive study and contributions on systems of multiple masses suspended and connected by light inextensible pieces of thread at small angles of inclination to the vertical by using Lagrange method has been done. We have been specific in varying masses and lengths and noted that by introducing additional mass in a system, the interaction coupling terms increase complicating the kinetic energy specifications.
Calculation of the energy explicitly for a general n-pendula system is derived. There are unique possibilities of advancing studies in n-tupple pendulum systems by Lagrange method to which we eagerly embraced. We have restricted our work to a series of n- pendulum system oscillating in a plane. The approach is pegged on the derivation of equations of motion using the energy in the systems and this general energy equation covers all combinations of varying masses, length and angles of inclinations. This study enables one to understand motions of cross coupled systems of any degrees of freedom at varying complexions of mechanical misalignment. This finds interesting applications in the study of various systems like describing the spin and orbital dynamics of super fluids. ^{3}He-A done by Sayev and Peletminsky (1997) where the equations of motion of the dynamic variables were derived using the Hamiltonian formalism.

THE ENERGY EQUATION

The Lagrangian formulation for n suspended masses m_i at distances of separation l_i and at angles of inclination θ_i to the vertical axis is first developed. The general resultant velocity component is calculated from the components of the effective displacement as follows:

$$\dot{x_n} = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + l_3\dot{\theta}_3\cos\theta_3 + \cdots + l_{n-2}\,\dot{\theta}_{n-2}\cos\theta_{n-2} + l_{n-1}\,\dot{\theta}_{n-1}\cos\theta_{n-1} + l_n\dot{\theta}_n\cos\theta_n \qquad (1)$$

$$\therefore \dot{y_n} = -\left\{\begin{matrix} l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 + l_3\dot{\theta}_3\sin\theta_3 + \cdots + l_{n-2}\,\dot{\theta}_{n-2}\sin\theta_{n-2} + \\ l_{n-1}\,\dot{\theta}_{n-1}\sin\theta_{n-1} + l_n\dot{\theta}_n\sin\theta_n \end{matrix}\right\} \qquad (2)$$

These equations lead to

$$(\dot{r_n})^2 = \left\{\begin{matrix} {l_1}^2{\dot{\theta}_1}^2 + {l_2}^2{\dot{\theta}_2}^2 + {l_3}^2{\dot{\theta}_3}^2 + \cdots + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \\ 2l_1l_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + \cdots + 2l_{n-2}l_{n-1}\dot{\theta}_{n-2}\dot{\theta}_{n-1}\cos(\theta_{n-1}-\theta_{n-2}) + \\ 2l_{n-1}l_n\dot{\theta}_{n-1}\dot{\theta}_n\cos(\theta_{n-1}-\theta_n) \end{matrix}\right\} \qquad (3)$$

The n^{th} kinetic energy term is determined as

$$T_n = m_k\left\{\frac{1}{2}\sum_{i=1}^{k} {l_i}^2\,{\dot{\theta}_i}^2 + \sum_{\substack{i=1\\ j\neq i}}^{k} l_il_j\,\dot{\theta}_i\dot{\theta}_j\cos(\theta_i-\theta_j)\right\} \qquad (4)$$

The total kinetic energy T_T is established by taking a summation overall all values of k.

$$T_T = \sum_{k=1}^{n} m_k\left\{\frac{1}{2}\sum_{i=1}^{k} {l_i}^2\,{\dot{\theta}_i}^2 + \sum_{\substack{i=1\\ j\neq i}}^{k} l_il_j\,\dot{\theta}_i\dot{\theta}_j\cos(\theta_i-\theta_j)\right\} \qquad (5)$$

Similarly, the total potential energy is calculated as

$$V_T = \sum_{k=1}^{n} m_k\left\{\left(\sum_{i=1}^{n} l_i\right) - \left(\sum_{i=1}^{k} l_i\cos\theta_i\right)\right\}g \qquad (6)$$

From Equations (5) and (6) we develop the Lagrangian for *n-tuple* pendula system as

$$L = \sum_{k=i}^{n} m_k \left\{ \frac{1}{2} \sum_{i=1}^{n} l_i^{\,2} \dot{\theta}_i^{\,2} + \sum_{\substack{i=1 \\ j \neq i}}^{n} l_i l_j \dot{\theta}_i \dot{\theta}_j \cos(\theta_i - \theta_j) \right\} - \sum_{k=i}^{n} m_k \left\{ \left(\sum_{i=1}^{n} l_i \right) - \left(\sum_{i=1}^{k} l_i \cos \theta_i \right) \right\} g \tag{7}$$

If $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = \frac{\partial L}{\partial \theta_i}$ then

$$\left(\sum_{k=f(i,j)}^{n} m_K\right)\left\{l_i^{\,2} \ddot{\theta}_i + l_i \sum_{j=1}^{n} l_j \ddot{\theta}_j\right\} = -(l_i \theta_i) g \left(\sum_{k=i}^{n} m_K\right) \tag{8}$$

for small $\dot{\theta}_i, \dot{\theta}_j$ $(k \geq i, j;\ j \neq i)$. Hence

$$l_i\left(\sum_{k=f(i,j)}^{n} m_K\right)\left(\sum_{j=1}^{n} l_j \ddot{\theta}_j\right) + \left(\sum_{k=i}^{n} m_K\right)\left\{l_i^{\,2} \ddot{\theta}_i\right\} = -(l_i \theta_i) g \left(\sum_{k=i}^{n} m_K\right) \tag{9}$$

Which can easily be used to generate a matrix for the *n-tupple* pendula systems for $n = \{1,2,3,4,5, \dots, n\}$. We can quickly examine the equation by fitting in values for simple cases. For a *Simple Pendulum*, *i = 1* hence from equation (9) we can easily work out the equation of motion as

$$m_1 l_1^{\,2} \ddot{\theta}_1 = -m_1 l_1 g \theta_1$$

$$\therefore \ddot{\theta}_1 = -{}^{g}\!/_{l_1}\, \theta_1 \tag{10}$$

whose particular solution is

$$\theta = \theta_0 \cos t \left(\frac{g}{l}\right)^{\frac{1}{2}} + \theta_1 \left(\frac{l}{g}\right)^{\frac{1}{2}} \sin t \left(\frac{g}{l}\right)^{\frac{1}{2}} \tag{11}$$

For a *Double Pendulum* with uneven quantities

a) $i = 1$ then

$$(m_1 + m_2) l_1^{\,2} \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 = -(m_1 + m_2) l_1 \theta_1 g \quad k \geq i, j \quad i \neq j \tag{12}$$

b) $i = 2$

$$l_2^{\,2} \ddot{\theta}_2 m_2 + m_2 l_1 l_2 \ddot{\theta}_1 = -m_2 l_2 \theta_2 g$$

From which we can write

$$\ddot{\theta}_1 = \left\{ \left(\frac{m_2}{m_1}\right) \theta_2 - \left(\frac{m_1 + m_2}{m_1}\right) \theta_1 \right\} \frac{g}{l_1} \tag{13}$$

$$\ddot{\theta}_2 = \left\{ \left(\frac{m_1 + m_2}{m_1}\right) (\theta_1 - \theta_2) \right\} \frac{g}{l_2} \tag{14}$$

A *double Pendulum* with even quantities (masses and lengths) gives

$$2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\theta_1 \left({}^{g}\!/_{l}\right) \tag{15}$$

whose solutions for θ_1 *and* θ_2 will be given by:-

$$\theta_1 = (0.5\theta_{10} + 0.3546\theta_{20}) \cos 0.76539t \left(\frac{g}{l}\right)^{\frac{1}{2}} + \left(\frac{l}{g}\right)^{\frac{1}{2}} (0.6483596\theta_{11} + 0.4589\theta_{21}) \sin 0.76539t \left(\frac{g}{l}\right)^{\frac{1}{2}} + (0.5\theta_{10} - 0.3546\theta_{20}) \cos 1.84742t \left(\frac{g}{l}\right)^{\frac{1}{2}} + \left(\frac{l}{g}\right)^{\frac{1}{2}} (0.268\theta_{11} - 0.1916\theta_{21}) \sin 1.84742t \left(\frac{g}{l}\right)^{\frac{1}{2}} \tag{16}$$

$$\theta_2 = (0.707\theta_{10} + 0.5015\theta_{20}) \cos 0.76539t \left(\frac{g}{l}\right)^{\frac{1}{2}} + \left(\frac{l}{g}\right)^{\frac{1}{2}} (0.92\theta_{11} + 0.649\theta_{21}) \sin 0.76539t \left(\frac{g}{l}\right)^{\frac{1}{2}} - (0.707\theta_{10} - 0.5015\theta_{20}) \cos 1.84742t \left(\frac{g}{l}\right)^{\frac{1}{2}} - \left(\frac{l}{g}\right)^{\frac{1}{2}} (0.37868\theta_{11} 0.2707\theta_{21}) \sin 1.84742t \left(\frac{g}{l}\right)^{\frac{1}{2}} \quad (17)$$

Similar computation can be done for higher combinations. We verified up to for a case of *5-tuple* pendula system. For the *n-tupple* pendulum systems we use the generalized energy equation to generate the equations as

$$\ddot{\boldsymbol{\theta}}_{(n-j)} = \left\{ \begin{matrix} \left(\frac{\sum_{i=(n-j-1)}^{n} m_i}{m_{(n-j-1)}}\right) \boldsymbol{\theta}_{(n-j-1)} - \left(\frac{m_{(n-j-1)} + m_{(n-j)}}{m_{(n-j-1)}}\right) \left(\frac{\sum_{(n-j)}^{n} m_i}{m_{(n-j)}}\right) \boldsymbol{\theta}_{(n-j)} + \\ \left(\frac{\sum_{(n-j+1)}^{n} m_i}{m_{(n-j)}}\right) \boldsymbol{\theta}_{(n-j+1)} \end{matrix} \right\} \frac{g}{l_{(n-j)}}$$

$$\forall m_i \neq m_j \quad n \geq 3, \quad n > j \quad (18)$$

$$\ddot{\boldsymbol{\theta}}_n = \left\{ \left(\frac{m_{(n-1)} + m_n}{m_{(n-1)}}\right) \left(\boldsymbol{\theta}_{(n-1)} - \boldsymbol{\theta}_n\right) \right\} \frac{g}{l_n} \quad (19)$$

DISCUSSION AND CONCLUSION

We have deduced that for equal masses and lengths the motion of any mass m_i depends in an implicit manner, on the motions of the nearest neighbours m_{i-1} and m_{i+1} other than itself. This pattern does not apply for the first and the last masses, which had only one neighbor each. The angular acceleration $\ddot{\theta}_i$ for any mass m_i depends upon its own angle θ_i and the angles of the nearest neighbours θ_{i-1} and θ_{i+1}. The chaotic nature increased as the links of the multiple pendulum system increased since more links increases more terms to the equations that form components of the Lagrangian causing the system to rely heavily on the initial conditions. It is evident that all the masses m_i below depended on all the length l_i above. Similarly all the masses m_j above are independent of all the length l_j below.

Since this work is restricted to motion in a plane (X, Y, O) for the Lagrangian Formulation, one may want to compare with the outcomes of possibly using Hamilton-Jacobi functions. It might be even more interesting to explore a 3-Dimension multi-pendula system.

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REFERENCES

Acebron JA, Bonilla LL Vicente CJP, Ritort F and Spigler R (2005). The Kuramoto Model: A simple Paradigm for Synchronization Phenomena. *Reviews of Modern Physics* 77**(**1) 137-185.

Chang KC, Long Y and Zehnder E (1990). Forced Oscilations for the Triple Pendulum. *Rabinowitz and E Zehnder Academic press, New York* 177-209

Dr fler F (2011). Synchronization ant Transient Stability in power and non-uniform Kuratoto Oscillators.

Furuta K, Yamakita M and Kobayashi S (1993). Swing –up Control of Inverted Pendulum using Pseudo-state Feedback. *Proceedings of the Institution of Mechanical Engineers* **206** *263-269.*

Giarratana J (1945). Equations of motion for classical dynamical Systems of Variable Mass. *Physical Review* **68(**5-6) 130-141

Koslov VV (1985). Calculus of Variation in the Large. *Russian Mathematical Survey* ***40*** *238-244.*

Kuramoto Y (2003). Chemical Oscillations, Waves and Turbulence. *Dover Publications.*

Laederich S and Levi M (1992*).* Qualitative Dynamics of Planar chains. *Physica D* ***54*** *(173-182).*

Prichan JS, Sakwa TW, Ayodo YK and Sarai A (2012). Equations of Motion for Multiple – Pendula System. *International Journal of Physics and Mathematical Sciences 2(2) in press.*

Sayev AA and Peletminsky SV (1997). On Hamiltonian Formulation of Hydrodynamic equations for Superfluids ^{3}He-A, *Low Temperature Physics*, **23(**12) 955-967

Strogatz SH (2000). From Kuramoto to Crawford: Exploring the onset of Synchronization in Populations of Coupled Oscillators. *Physica D: Nonlinear Phenomena* **143**(1) 1-20

Tang CL (2005), Fundamentals of Quantum Mechanics For Solid State Electronics and Optics, *Cambridge University Press, New York.* **1** 8-25

White W, Niemann D, Lynch M (1989). The Presentation of Lagranges Equations in Introductory Robotics Course. *IEEE Transactions on Education*, New York, USA **32(**1) 39-46.