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Research Article

EQUATIONS OF MOTION FOR MULTIPLE – PENDULA SYSTEM

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ABSTRACT
A four and five-tupple chain pendulum system constrained to move in a plane(X, Y,0) has been studied within the framework of a generalized coordinate system by using an abridged Lagrangian formalism. The Lagrangian is developed and used to get the equations of motion which are further solved using eigenvalue relations. In an ideal case of a non-viscous motion, a series of combinations of pendulum units remains a salient feature of this work. As a rule, it is generally observed that the angular acceleration for any mass is influenced by the masses and angles of the immediate neighbour-masses.

***Key Words**: Lagrangian, Generalized Coordinates*

INTRODUCTION
The study of the equations of motion for *n-tupple* pendulum systems can be explored using Newtonian mechanics. This is extensively used for the dynamics of masses in the determination of the dynamical variables. It is the limitation of the Newtonian approach that led to the introduction of the use of the energy dependent Lagrangian approach. Generalized co-ordinates formalism exhibits domino effects to easy evaluation of various quantities like velocities, forces and momenta. Attempts have been made by scholars to obtained equations of motion and their solutions for multiple pendulum systems using the Lagrangian formulation (Spiegel, 1967; Chow, 1995). There are limited initiatives to advance studies of motions of many-body pendulum system (*n-tupple* pendula System, where $n \geq 3$) fixed at one end and connected in series, one after another via an inextensible string. The approach of the generalized co-ordinates depends on the angle of inclications, $\theta_{i,}$ in two dimensions.
There has been an extensive study of coupled systems as new qualitative systems save for the theoretical challenges involved in the investigations of such systems (Hedrih, 1999, 2007). Free vibrations of a multi-pendulum system inter-coupled by standard light elements and different properties have been explored in which the obtained analytical solutions are numerically analyzed (Hedrih, 2008). Annotations have been made on counter – intuitive phenomenon of a driven inverted chain consisting of N linked pendulums balanced on top of one another in which the amplitudes diminishes with the number of pendulums involved in the chain (Acheson, 1993, 2005; Acheson *et al.,* 1993).
Numerous extensions of pendulum systems have been proposed and studied which includes various categories of elastic pendulum models and multi-body pendulum models (Furuta *et al.,* 1993, 1984; Spong *et al.,* 2001). The initial conditions in the agreement protocol can be manipulated to produce results that satisfy linear constraints and the same can apply to the control of a distributed network of linearized pendula on a line graph topology (Nedic *et al.,* 2008).
At the outset of this study, Spiegel (1969) worked on two mass pendulum system in which the masses as well as their lengths of separation were equal in magnitude. Chow (1995) handled a double pendulum with uneven quantities using the Lagrange method. Jones *et al.,* (2011) gave more information about the examination of chaos variation in multiple pendulum systems with different amounts of energy. They strongly speculated that for varying masses and lengths, the problem becomes complex with increase in mass units. Furuta *et al.,* (1993, 1984), successfully worked on the control of the multiple pendulums of varying constraints and further discussed the swing-up control of pendulum units by considering the reach-ability of an unstable nonlinear control. Joot (2009) noted that by introducing additional mass in a system, the interaction coupling terms increases thus complicating the kinetic energy specifications noting that calculating the energy explicitly for a general n-pendula system is likely to be pedantic for even the

most punishing instructor to inflict on students as a problem. This statement is what basically triggered our curiosity and forms the basis of our work.
We have been able to give a quantitative study of a multiple pendula system through rigorous exercises. Unnoticeable work may have been done on these systems but not for mass units more than three. There are unique possibilities of advancing studies in n-tupple pendulum systems by Lagrange method to which we eagerly are embracing. A comprehensive study and contributions on systems of four and five masses suspended and connected in series by light inextensible pieces of thread at relatively small angles of inclination to the vertical by using Lagrange method has been done. This work covers all the aspects of suspended masses in series for four and five pendulum systems using the Lagrange method. We have restricted our work to a series of multiple- pendulum system oscillating in a plane. The approach is pegged on the derivation of equations of motion using the energy in the systems. We deviate from the traditional classical mechanics method that is anchored on the concept of force which becomes more complex even for a few masses. Apart from its intrinsic utility as a timing device, the pendulum is a superb learning tool for science education. It serves as a model for study of the linear oscillators.

THEORETICAL DERIVATIONS

The Lagrangian was developed to simplify the achievement of solutions to equations of motion using the difference in the kinetic energy and the potential energy for such bodies in motion. In this approach we need to find the degrees of freedom that would fit the situation and thus give a general view point. Generally force can be related to the kinetic energy by the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=\Phi_i \qquad (1)$$

where $L=T-V$ is the Lagrangian and q_i & $\dot{q}_i$ are the generalized coordinates for position and momentum respectively. For conservative systems, $\Phi_i = 0$, reducing equation (1) to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0 \qquad (2)$$

Equations of motion depends on the degrees of freedom hence the need to use generalized Lagrangian so as to be applicable to as many masses as possible. This has been done in order to take care of the possibility of having various combinations of equal or unequal masses, lengths and angles of inclination to the vertical.

The kinetic energy and potential energy terms for three masses have been calculated to be as given in equations (3) and (4) respectively as shown below.

$$T = \tfrac{1}{2}m_1\left(l_1{}^2\dot{\theta}_1{}^2\sin^2\theta_1 + l_1{}^2\dot{\theta}_1{}^2\cos^2\theta_1\right) + \tfrac{1}{2}m_2\left\{\left(l_1{}^2\dot{\theta}_1{}^2\sin^2\theta_1 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + l_2{}^2\dot{\theta}_2{}^2\sin^2\theta_2\right) + \left(l_1{}^2\dot{\theta}_1{}^2\cos^2\theta_1 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + l_2{}^2\dot{\theta}_2{}^2\cos^2\theta_2\right)\right\} + \tfrac{1}{2}m_3\left\{\left(l_1{}^2\dot{\theta}_1{}^2\sin^2\theta_1 + l_2{}^2\dot{\theta}_2{}^2\sin^2\theta_2 + l_3{}^2\dot{\theta}_3{}^2\sin^2\theta_3 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + 2l_1l_3\dot{\theta}_1\dot{\theta}_3\sin\theta_1\sin\theta_3 + 2l_2l_3\dot{\theta}_2\dot{\theta}_3\sin\theta_2\sin\theta_3\right) + \left(l_1{}^2\dot{\theta}_1{}^2\cos^2\theta_1 + l_2{}^2\dot{\theta}_2{}^2\cos^2\theta_2 + l_3{}^2\dot{\theta}_3{}^2\cos^2\theta_3 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + 2l_1l_3\dot{\theta}_1\dot{\theta}_3\cos\theta_1\cos\theta_3 + 2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos\theta_2\cos\theta_3\right)\right\} \qquad (3)$$

$$V = m_1g\{l_1 + l_2 + l_3 - l_1\cos\theta_1\} + m_2g\{l_1 + l_2 + l_3 - (l_1\cos\theta_1 + l_2\cos\theta_2)\} + m_3g\{l_1 + l_2 + l_3 - (l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3)\} \qquad (4)$$

where m_i is the i^{th} mass, l_i is the i^{th} length and θ_i is the i^{th} angle of inclination. Equations (3) and (4) give the Lagragian as

$$L = \tfrac{1}{2}m_1{l_1}^2{\dot\theta_1}^2 + \tfrac{1}{2}m_2\left\{{l_1}^2{\dot\theta_1}^2 + {l_2}^2{\dot\theta_2}^2 + 2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right\} + \tfrac{1}{2}m_3\left\{{l_1}^2{\dot\theta_1}^2 + {l_2}^2{\dot\theta_2}^2 + {l_3}^2{\dot\theta_3}^2 + 2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2) + 2l_1l_3\dot\theta_1\dot\theta_3\cos(\theta_1-\theta_3) + 2l_2l_3\dot\theta_2\dot\theta_3\cos(\theta_2-\theta_3)\right\} - m_1g\{l_1+l_2+l_3-l_1\cos\theta_1\} - m_2g\{l_1+l_2+l_3-(l_1\cos\theta_1+l_2\cos\theta_2)\} - m_3g\{l_1+l_2+l_3-(l_1\cos\theta_1+l_2\cos\theta_2+l_3\cos\theta_3)\} \quad (5)$$

The Lagragian used in equation (2) generates equations of motion for three uneven quantities as given in equations (6), (7) and (8);

$$\ddot\theta_1 = \frac{1}{(m_1l_1)}\{(m_2+m_3)\theta_2-(m_1+m_2+m_3)\theta_1\}g \quad (6)$$

$$\ddot\theta_2 = \frac{1}{m_2l_2}\left\{\frac{m_2}{m_1}(m_1+m_2+m_3)\theta_1 - \frac{(m_1+m_2)(m_2+m_3)}{m_1}\theta_2 + m_3\theta_3\right\}g \quad (7)$$

$$\ddot\theta_3 = \frac{1}{m_2l_3}\{(m_2+m_3)(\theta_2-\theta_3)\}g \quad (8)$$

A special case of equal masses and distances gives the following equations

$$\ddot\theta_1 = (2\theta_2-3\theta_1)\tfrac{g}{l} \quad (9)$$

$$\ddot\theta_2 = (3\theta_1-4\theta_2+\theta_3)\tfrac{g}{l} \quad (10)$$

$$\ddot\theta_3 = 2(\theta_2-\theta_3)\tfrac{g}{l} \quad (11)$$

Eigen formulae in equations (12), (13) and (14) were then used to get the solutions of the equations of motion.

$$\lambda + 1 = 0 \quad \text{(for one mass)} \quad (12)$$

$$2\lambda^2 + 4\lambda + 1 = 0 \quad \text{(for two equal masses)} \quad (13)$$

$$6\lambda^3 + 18\lambda^2 + 9\lambda + 1 = 0 \quad \text{(for three equal masses)} \quad (14)$$

Similar calculations, for special cases of equal masses and lengths, were done to get the following sets of equations of motion for four and five masses respectively

$$\begin{aligned} 4\ddot\theta_1 + 3\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4 &= -4\theta_1 \, g/l \\ 3\ddot\theta_1 + 3\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4 &= -3\theta_2 \, g/l \\ 2\ddot\theta_1 + 2\ddot\theta_2 + 2\ddot\theta_3 + \ddot\theta_4 &= -2\theta_3 \, g/l \\ \ddot\theta_1 + \ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4 &= -\theta_4 \, g/l \end{aligned} \quad (15)$$

$$\begin{aligned} 5\ddot\theta_1 + 4\ddot\theta_2 + 3\ddot\theta_3 + 2\ddot\theta_4 + \ddot\theta_5 &= -5\theta_1 \, g/l \\ 4\ddot\theta_1 + 4\ddot\theta_2 + 3\ddot\theta_3 + 2\ddot\theta_4 + \ddot\theta_5 &= -4\theta_2 \, g/l \\ 3\ddot\theta_1 + 3\ddot\theta_2 + 3\ddot\theta_3 + 2\ddot\theta_4 + \ddot\theta_5 &= -3\theta_3 \, g/l \\ 2\ddot\theta_1 + 2\ddot\theta_2 + 2\ddot\theta_3 + 2\ddot\theta_4 + \ddot\theta_5 &= -2\theta_4 \, g/l \\ \ddot\theta_1 + \ddot\theta_2 + \ddot\theta_3 + \ddot\theta_4 + \ddot\theta_5 &= -\theta_5 \, g/l \end{aligned} \quad (16)$$

DISCUSSION AND CONCLUSION

We deduce that$(where\ m_i = m_j;\ l_i = l_j;\ \theta_i \neq \theta_j)$, the motion of any mass m_i depends in an implicit manner, on the motions of the nearest neighbours m_{i-1} and m_{i+1} other than itself. This pattern does not apply for the first and the last masses, which had only one neighbor each. The angular acceleration $\ddot\theta_i$ for any mass m_i will depend upon its own angle θ_i and the angles of the nearest neighbours θ_{i-1} and θ_{i+1}. The chaotic nature increases as the links of the multiple pendulum system increases. This is because more links increases more terms to the equations that form components of the Lagrangian causing the system to rely heavily on the initial conditions. It will be interesting to develop equations for all combinations of varying masses, length and angles of inclinations. These may easily establish a pattern that can enable one to get equations of motion for ***n-tupple*** pendulum system. This study enables one to understand motions of cross coupled systems of any degrees of freedom at varying complexions of mechanical misalignment. We intend to make a communication on a more generalized n-tupple pendulum system as soon as an acceptable model is developed. This may particularly be more appreciable in the study of central force systems.

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