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ON THE EXTENSION OF THERMONUCLEAR FUNCTIONS THROUGH THE PATHWAY MODEL INCLUDING MAXWELL-BOLTZMANN AND TSALLIS DISTRIBUTIONS AND $\overline{H}$ -FUNCTION

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ABSTRACT
The Maxwell-Boltzmannian approach to nuclear reaction rate theory is extended to cover Tsallis statistics (Tsallis 1988) and more general cases of distribution functions. An analytical study of respective thermonuclear functions is being conducted with the help of statistical techniques. The pathway model, recently introduced by Mathai (1993), is utilized for thermonuclear functions and closed-form representations are obtained in terms of $\overline{H}$ -functions and G -functions. Maxwell-Boltzmannian thermonuclear functions become particular cases of the extended thermonuclear functions. A brief review on the development of the theory of analytic representations of nuclear reaction rates is given.

INTRODUCTION
In the evolution of the Universe, chemical elements are created in cosmological and stellar nucleosynthesis (Clayton 1983 and Fowler 1984). Solar nuclear energy generation and solarneutrino emission are governed by chains of nuclear reactions in the gravitationally stabilized solar fusion reactor (Davis 2003). One of the first utilization of Gamow's quantum mechanical theory of potential barrier penetration to other than the analysis of radioactive decay was to the question on how do stars generate energy (Critchfield 1972) for a brief essay on the history of this discovery see Mathai and Haubold (1998). Continued attempts are aiming at generating energy through controlled thermonuclear fusion in the laboratory. In nuclear plasma, the rate of reactions and thus energy releases can be determined by an average of the Gamow penetration factor over the distribution of velocities of the particles of the plasma (Haubold and Mathai 1984). Understanding the mathematical and statistical methods for the evaluation of thermonuclear reaction rates is one of the goals of research in the field of stellar and cosmological nucleosynthesis. Practically all applications of fusion plasmas are controlled in some way or another by the theory of thermonuclear reaction rates under specific circumferences. After several decades of effort, a systematic and complete theory of thermonuclear reaction rates has been developed (Haubold and John 1978, Anderson *et al.*, 1994, Haubold and Mathai 1984, Mathai and Haubold 1973). Reactions between individual particles produce a randomization of the energy and velocity distributions of particles. The depletion of particles by reactions is balanced by the diffusion of the particles in the macroscopically inhomogeneous medium. As a result of this balance, the fusion plasma may reach a quasi-stationary state close to equilibrium, in which steady of matter, energy, and momentum are present. This also led to the assumption that the distribution of particles can be assumed to be Maxwell-Boltzmannian in almost all cases of interest to stellar physics and cosmology.
The derivation of closed-form representations of nuclear reaction rates and useful approximations of them are based on statistical distribution theory and the theory of generalized special functions, mainly in the categories of Meijer's G -function and Fox's $\overline{H}$ -function. For an overview on the application and historical background of integrals and distribution functions for reaction rates and their representation in terms of special functions, see Hegyi (1999) and Moll (2002). Special cases which can be derived from the general theory through expansion of respective physical parameters (like the cross section factor) are resonant (Hussein and Pato 1997, Ueda *et al.,* 2001) and resonant (Udea *et al.,* 2004, Newton *et al.,* 2007)

reaction rates, reaction rates for cosmological nucleosynthesis (Bergstroem *et al.*, 1999) and fitting of experimental data to analytic representations (Brown and Jarmie 1998). Specify mathematical methods for deriving approximate analytic representations of nuclear reaction rates are expansions of hypergeometric functions (Saigo and Saxena 1998), transformation of extended gamma functions (Aslam Chaudhary 2002), and asymptotic expansion of the Laplace transform of functions (Ferreira and Lopez 2004).

Only recently, related to the production of neutrinos in the gravitationally stabilized solar fusion reactor, the question of possible derivations of the velocity distribution of plasma particles from the Maxwell-Boltzmannian case due to memory effects and long-range forces has been raised (Coraddu *et al.*, 1999, Lavagno and Quarati 2002, Coraddu *et al.*, 2003, Lissia and Quarati 2005, Lavagno and Quarati 2006). This was initiated by Tsallis non-additive generalization of Boltzmann-Gibbs statistical mechanics which generates q-exponential function as the fundamental distribution function instead of the Maxwell-Boltzmann distribution function. Tsallis statistics covers Boltzmann-Gibbs statistics for the case $q=1$ (Tsallis 1988, Gell-Mann and Tsallis 2004, Tsallis 1988). This paper develops the complete theory for closed-form representations of nuclear reaction rates for Tsallis statistics. In this context an interesting discovery was made by Mathai (1993), namely that even more general distribution functions can be incorporated in the theory of nuclear reaction rates by appealing to entropic and distributional pathways. Starting from generalized entropy of order α, through the maximum entropy principle, distribution functions are generated which include Maxwell-Boltzmann and Tsallis distributions as special cases.

In subsections 1.1, 1.2, 1.3 and 1.4, we introduce the definition of the thermonuclear reaction rate and respective thermonuclear functions for the standard, cut-off, depleted, and screened case, respectively. Each subsection provides the integral of the thermonuclear function for the cases of Maxwell-Boltzmann distribution and α distribution, the latter covers the q-exponential of Tsallis. Section 2 provides prerequisites for the use of G - and $\overline{H}$ - functions in Mellin-Barnes integral representation and also discusses briefly the pathway model of Mathai. Section 3 elaborates the closed-form representation of the thermonuclear functions for the case of Maxwell-Boltzmann and α distributions in terms of G - and $\overline{H}$ - functions. Section 4 provides conclusions.

From (Mathai and Haubold 1998) it can be seen that the expression for the reaction rate r_{ij} of the reacting particles i and j taking place in a nondegenerate environment is

$$\begin{aligned} r_{ij} &= n_i n_j \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{kT}\right)^{3/2} \int_0^\infty E\,\sigma(E)\, e^{-\frac{E}{kT}} dE \\ &= n_i n_j \langle \sigma v \rangle \end{aligned} \tag{1.1}$$

Where n_i and n_j are the number densities of the particles i and j, the reduced mass of the particles is denoted by $\mu = \dfrac{m_i m_j}{m_i + m_j}$, T is the temperature, k is the Boltzmann constant, the reaction cross section is $\sigma(E)$ and the kinetic energy of the particles in the center of mass system is $E = \dfrac{\mu v^2}{2}$ where v is the relative velocity of the interacting particles i and j .

The reaction probability is written in the form $\langle \sigma v \rangle$ to indicate that it is an appropriate average of the product of the reaction cross section and relative velocity of the interacting particles. For detailed physical interpretations see Haubold and Mathai (1984).

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Standard Non-Resonant Thermonuclear Function

For non-resonant nuclear reactions between two nuclei of charges z_i and z_j colliding at low energies below the Coulomb barrier, the reaction cross section has the form Mathai and Haubold (1998)

$$\sigma(E) = \frac{S(E)}{E} e^{-2\pi\eta(E)}$$

With $$\eta(E) = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{z_i z_j e^2}{hE^{\frac{1}{2}}}$$

Where $\eta(E)$ is the Sommerfeld parameter, $\hbar$ is the Planck's quantum of action, e is the quantum of electric charge, the cross section factor $S(E)$ is often found to be a constant or a slowly varying function of energy over a limited range of energy (Mathai and Haubold 1998). The cross section factor $S(E)$ may be expressed in terms of the power series expansion,

$$S(E) = S(0) + \frac{dS(0)}{dE} E + \frac{1}{2}\frac{d^2 S(0)}{dE^2} E^2$$

Then

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{v=0}^{2} \left(\frac{1}{kT}\right)^{-v+\frac{1}{2}} \frac{S^{(v)}(0)}{v!} \int_0^\infty E^v e^{-\frac{E}{kT} - 2\pi\eta(E)} dE$$

$$= \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{v=0}^{2} \left(\frac{1}{kT}\right)^{-v+\frac{1}{2}} \frac{S^{(v)}(0)}{v!} \int_0^\infty x^v e^{-x - bx^{-\frac{1}{2}}} dx \qquad (1.2)$$

Where $x = \dfrac{E}{kT}$ and $b = \left(\dfrac{\mu}{2kT}\right)^{\frac{1}{2}} \dfrac{z_i z_j e^2}{\hbar}$

The collision probability integral, called thermonuclear function, for non-resonant thermonuclear reactions in the Maxwell-Boltzmannian case is Haubold and Mathai (1984)

$$I_1\left(v, 1, b, \frac{1}{2}\right) = \int_0^\infty x^v e^{-x - bx^{-\frac{1}{2}}} dx \qquad (1.3)$$

We will consider here the general integral

$$I_1(\gamma - 1, a, b, \rho) = \int_0^\infty x^{\gamma-1} e^{-ax - bx^{-\rho}} dx,\ a < 0, b > 0, \gamma > 0, \rho > 0 \qquad (1.4)$$

Non-Resonant Thermonuclear Function with High Energy Cut-Off

Usually, the thermonuclear fusion plasma is assumed to be in thermodynamic equilibrium. But if there appears a cut-off of the high energy tail of the Maxwell-Boltzmann distribution function in (1.3) then the thermonuclear function is given by

$$I_2^{(d)}\left(v, 1, b, \frac{1}{2}\right) = \int_0^d x^v e^{-x - bx^{-\frac{1}{2}}} dx,\ b > 0, \gamma > 0, v > 0 \qquad (1.5)$$

Again we consider the general form of the integral (1.5) as

$$I_2^{(d)}(\gamma - 1, a, b, \rho) = \int_0^d x^{\gamma-1} e^{-x - bx^{-\rho}} dx,\ a > 0, b > 0, \gamma > 0, \rho > 0 \qquad (1.6)$$

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For physical reasons for the cut-off modification of the Maxwell-Boltzmann distribution function of the relative kinetic energy of the reacting particles refer to the paper (Haubold and Mathai 1984).

Non-Resonant Thermonuclear Function with Depleted Tail

A depletion of the high energy tail of the Maxwell-Boltzmann distribution function of the relative kinetic energies of the niclei of the fusion plasma is discussed in Haubold and Mathai (1984).

For the thermonuclear function, in comparison to the strict Maxwell-Boltzmannian case, we have the integral

$$I_3\left(\nu,1,1,\delta,b,\frac{1}{2}\right)=\int_0^\infty x^{\nu}e^{-x-x^{\delta}-bx^{\frac{1}{2}}}dx\,,b>0,\delta>0,\nu>0 \tag{1.7}$$

We will consider the general integral of the type

$$I_2^{(d)}\left(\gamma-1,a,z,\delta,b,\rho\right)=\int_0^d x^{\gamma-1}e^{-ax-zx^{\delta}-bx^{-\rho}}dx \tag{1.8}$$

Where $z>0,a>0,b>0,\rho>0$.

Non-Resonant Thermonuclear Function with Screening

The plasma correction to the fusion process due to a static or dynamic potential, i.e. .The electron screening effects for the reacting particles, the collision probability integral to be evaluated in the case of the screened non-resonant nuclear reaction rate is see Haubold and Mathai (1984)

$$I_4\left(\nu,1,b,t,\rho\right)=\int_0^\infty x^{\nu}e^{-x-b(x+t)^{\frac{1}{2}}}dx\,,\ b>0,\delta>0,\nu>0,\rho>0 \tag{1.9}$$

In this case we consider the general integral as

$$I_4\left(\gamma-1,a,b,t,\rho\right)=\int_0^\infty x^{\gamma-1}e^{-ax-b(x+t)^{\frac{1}{2}}}dx\,,\ b>0,a>0,t>0,\rho>0 \tag{1.10}$$

Where t is the electron screening parameter.

In the following, we are evaluating the thermonuclear reaction probability integrals by a using pathway model (Mathai 1993). In section 2 we give the basic definitions that we require in this paper. We evaluate the integral I_1 and I_2 by implementing Mathai's pathway model and represent each of them in terms of $\overline{H}$ -function and G -function in section 3.

Mathematical Preliminaries

We need some basic quantities for our discussion, which will be defined here. The Gamma function denoted by $\Gamma(z)$ for complex number z is defined as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\quad \mathrm{Re}(z)>0 \tag{2.1}$$

Where $\mathrm{Re}(.)$ denotes the real part of (.). In general $\Gamma(z)$ exists for all values of z , positive or negative, except at the point $z=0,1,2,\ldots$. These are the poles of $\Gamma(z)$. But the integral representation holds for the real part of z to be positive. Another important result we need is the multiplication formula. If z is any complex number, $z=0,1,2,\ldots$ and let m be a positive integer then the multiplication formula for Gamma functions is

$$\Gamma(mz)=(2\pi)^{\frac{1-m}{2}}m^{mz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right) \tag{2.2}$$

For $m=2$ we get the duplication formula for Gamma functions,

$$\Gamma(2z)=\pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{2.3}$$

The Mellin transform of a real scalar function $f(x)$ with parameter s is defined as

$$M_f(s)=\int_0^\infty x^{s-1}f(x)dx \tag{2.4}$$

Whenever $M_f(s)$ exists. If $f_1(x)$ and $f_2(y)$ are integrable functions on the positive real and if $x^k f_1(x)$ and $y^k f_2(y)$ are absolutely integrable, then the Mellin convolution property is defined as

$$f_3(u)=\int_0^\infty \frac{1}{x}f_1(x)f_2\left(\frac{y}{x}\right)dx \tag{2.5}$$

Then the Mellin transform of f_3 denoted by $M_{f_3}(s)$ is

$$M_{f_3}(s)=M_{f_1}(s)M_{f_2}(s) \tag{2.6}$$

Where $M_{f_1}(s)=\int_0^\infty x^{s-1}f_1(x)dx$ and $M_{f_2}(s)=\int_0^\infty x^{s-1}f_2(y)dy$.

The type-1 Beta integral is defined as

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}dy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.7}$$

$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$.

The type-2 Beta integral is defined as

$$\int_0^1 x^{\alpha-1}(1+x)^{-(\alpha+\beta)}dx=\int_0^1 y^{\beta-1}(1+y)^{-(\alpha+\beta)}dy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.8}$$

$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$.

The G -function which is originally due to see Meijer (1936), Mathai (1993) Mathai and Saxena (1973) is defined as a Mellin-Barnes type integral as follows:

$$G_{p,q}^{m,n}\left(z\left|{}_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right.\right)=\frac{1}{2\pi i}\int_L g(s)z^{-s}ds \tag{2.9}$$

Where $i=\sqrt{-1}, L$ is a suitable contour and $z\neq 0, m, n, p, q$ are integers, $0\leq m\leq q$ and $0\leq n\leq p$,

$$g(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+s\right)} \tag{2.10}$$

The $\overline{H}$ -function is defined in terms of a Mellin-Barnes type integral as

$$\overline{H}_{P,Q}^{M,N}[z]=\overline{H}_{P,Q}^{M,N}\left[z\left|{}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{2.11}$$

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Where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{2.12}$$

For further details of $\bar{H}$ -function, we refer to the original paper of Buschman and Srivastava (1990). Next we utilize the pathway model of Mathai (1993). When fitting a model to experimental data very often one needs a model for a distribution function from a given parametric family, or sometimes we may have a situation of the right tail cut-off. In order to take care of these situations and going from one functional from to another, a pathway parameter α is introduced, see Mathai [1993] and Mathai and Haubold [2007]. By this model we can proceed from a generalized type-1 beta model to a generalized type-2 beta model to a generalized gamma model when the variable is restricted to be positive. For the real scalar case the pathway model is the following,

$$f(x)=c\,|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{1}{1-\alpha}},a>0,\delta>0,1-a(1-\alpha)|x|^{\delta}>0,\gamma>0 \tag{2.13}$$

Where c is the normalizing constant and α is the pathway parameter. When $\alpha<1$ the model becomes a generalized type-1 beta model in the real case. This is a model with the right tail cut-off. When $\alpha>1$ we have $1-\alpha=-(\alpha-1),\alpha>1$ so that

$$f(x)=c\,|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{-\frac{1}{\alpha-1}}, \tag{2.14}$$

Which is a generalized type-2 beta model for real x . When $\alpha\to1$ the above two forms will reduce to

$$f(x)=c\,|x|^{\gamma-1}e^{-ax^{\delta}} \tag{2.15}$$

Observe that the normalizing constant c appearing in (2.13), (2.14) and (2.15), are different.

Evaluation Of The Integrals Of Thermonuclear Functions

Now we will evaluate the integrals I_1 and I_2 by introducing the pathway model.

Evaluation of I_1

$$I_1=\int_0^{\infty}x^{\gamma-1}e^{-ax-bx^{-\rho}}dx,\ a>0,b>0,\gamma>0,\rho>0$$

Replace e^{-ax} by $[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}$. As $\alpha\to1,[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}$ becomes e^{-ax} so that we can extend the integral I_1 to a wide class through the pathway parameter α. Let us denote the wider class of this integral by $I_{1\alpha}$.

$$I_{1\alpha}=\int_0^{\infty}x^{\gamma-1}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}e^{-bx^{-\rho}}dx$$

Where $\alpha>1,a>0,\delta=1,1+a(\alpha-1)x>0,\rho>0,b>0$. This is the product of two integrable functions. Hence we can apply Mellin convolution property for finding the value of the integral. Here let us take

$$f_1(x)=\begin{cases}x^{\gamma}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}} & for\ 0\le x<\infty,a>0,\gamma>0,\alpha>1\\ 0, & otherwise\end{cases} \tag{3.1}$$

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$$f_2(y)=\begin{cases} e^{-y^{\rho}} & \text{for } 0\le y<\infty,\rho>0 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

From (2.5) we have

$$I_{2\alpha}^{(d)}=\int_0^{\infty}\frac{1}{x}f_1(x)f_2\left(\frac{y}{x}\right)dx$$

$$=\int_0^{\infty}x^{\gamma-1}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}e^{-u^{\rho}x^{-\rho}}dx$$

$$=\int_0^{d}x^{\gamma-1}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}e^{-bx^{-\rho}}dx \quad where\ b=u^{\rho},\alpha>1 \tag{3.3}$$

The Mellin transform of $f_3=I_{1\alpha}$ is the product of the Mellin transforms of f_1 and f_2

$$M_{f_3}(s)=M_{f_1}(s)M_{f_2}(s)$$

$$M_{f_1}(s)=\int_0^{\infty}x^{s-1}x^{\gamma}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}dx$$

$$=\int_0^{\infty}x^{\gamma+s-1}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}dx$$

Putting $a(\alpha-1)x=t$, we get

$$M_{f_1}(s)=\frac{1}{[a(\alpha-1)]^{\gamma+s}}\int_0^{\infty}t^{\gamma+s-1}(1+t)^{-\frac{1}{\alpha-1}}dt$$

$$=\frac{1}{[a(\alpha-1)]^{\gamma+s}}\frac{\Gamma(\gamma+s)\Gamma\left(\frac{1}{\alpha-1}-\gamma-s\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)} \tag{3.4}$$

Where $\mathrm{Re}(\gamma+s)>0,\mathrm{Re}\left(\frac{1}{\alpha-1}-\gamma-s\right)>0,\alpha>1.$

$$M_{f_2}(s)=\int_0^{\infty}y^{s-1}e^{-y^{\rho}}dy$$

Putting $y^{\rho}=t$, we get

$$M_{f_2}(s)=\frac{1}{\rho}\int_0^{\infty}t^{\frac{s}{\rho}-1}e^{-t}dt$$

$$=\frac{1}{\rho}\Gamma\left(\frac{s}{\rho}\right),\quad \mathrm{Re}(s)>0 \tag{3.5}$$

From (3.4) and (3.5)

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$$M_{f_3}(s) = \frac{1}{\rho[a(\alpha-1)]^{\gamma+s}} \frac{\Gamma(\gamma+s)\Gamma\left(\frac{1}{\alpha-1}-\gamma-s\right)\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)}$$

Then the density of u denoted by $f_3(u)$ is available from the inverse Mellin transform.

$$f_3(u) = \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\gamma+s)\Gamma\left(\frac{1}{\alpha-1}-\gamma-s\right)\Gamma\left(\frac{s}{\rho}\right)$$

$$\times[a(\alpha-1)u]^{-s}\,ds \tag{3.6}$$

Comparing (3.3) and (3.6)

$$\int_0^\infty x^{\gamma-1}[1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}}e^{-bx^{-\rho}}dx$$

$$= \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} \overline{H}^{2,1}_{1,2}\left[a(\alpha-1)b^{\frac{1}{\rho}}\middle|\begin{matrix}\left(1-\frac{1}{\alpha-1}+\gamma,1;1\right)\\(\gamma,1),\left(0,\frac{1}{\rho}\right)\end{matrix}\right], \alpha>1$$

Therefore,

$$I_{1\alpha} = \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} \overline{H}^{2,1}_{1,2}\left[a(\alpha-1)b^{\frac{1}{\rho}}\middle|\begin{matrix}\left(1-\frac{1}{\alpha-1}+\gamma,1;1\right)\\(\gamma,1),\left(0,\frac{1}{\rho}\right)\end{matrix}\right], \alpha>1$$

Where $a>0, b>0, \gamma>0, \rho>0, \alpha>1, \mathrm{Re}(s)>0, \mathrm{Re}(\gamma+s)>0$. Note that when $\alpha\to 1, I_{1\alpha}$ becomes I_1. But $I_{1\alpha}$ contains all neighborhood solutions for various values of α for $\alpha>1$.

Special Cases

If $\frac{1}{\rho}$ is an integer then put $\frac{1}{\rho}=m$. Using equation (2.2) we get

$$I_{1\alpha} = \frac{\sqrt{m}\,(2\pi)^{\frac{1-m}{2}}}{[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} \overline{G}^{m+1,1}_{1,m+1}\left[\frac{a(\alpha-1)b^m}{m^m}\middle|\begin{matrix}\left(1-\frac{1}{\alpha-1}+\gamma;1\right)\\0,\frac{1}{m},\ldots,\frac{m-1}{m},\gamma\end{matrix}\right]$$

Where $a>0, b>0, \gamma>0, \alpha>1$.

In the thermonuclear function for a non-resonant thermonuclear reaction in the Maxwell-Boltzmannian case $\gamma-1=\nu, a=1, \rho=\frac{1}{2}$, then by using (2.3) we get,

$$I_{1\alpha} = \frac{1}{[a(\alpha-1)]^{\nu+1}\Gamma\left(\frac{1}{\alpha-1}\right)} \overline{G}^{3,1}_{1,3}\left[\frac{(\alpha-1)b^2}{4}\middle|\begin{matrix}\left(1-\frac{1}{\alpha-1}+\nu+1;1\right)\\0,\frac{1}{2},\nu+1\end{matrix}\right]$$

Where $b>0, \nu>0, \alpha>1$.

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Evaluation Of I_2

Replace e^{-ax} by $\left[1-a(1-\alpha)x\right]^{\frac{1}{1-\alpha}}$. As $\alpha \to 1, \left[1-a(1-\alpha)x\right]^{\frac{1}{1-\alpha}}$ becomes e^{-ax}. Let us denote the two integrals by $I_2^{(d)}$ and $I_{2\alpha}^{(d)}$ respectively.

$$I_2^{(d)} = \int_0^d x^{\gamma-1} e^{-ax-bx^{-\rho}} dx,\ a>0, b>0, \gamma>0, \rho>0$$

$$I_{2\alpha}^{(d)} = \int_0^d x^{\gamma-1}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-bx^{-\rho}} dx$$

Where $d \le \dfrac{1}{a(1-\alpha)}, \alpha<1, a>0, \delta=1, 1-a(1-\alpha)x>0, \rho>0, b>0$ .For convenience of integration let us assume that $d=\dfrac{1}{a(1-\alpha)}$. Then $I_{2\alpha}^{(d)}$ is the product of two integrable functions. Hence we can apply Mellin convolution property for finding the value of the integral. Here let us take

$$f_1(x) = \begin{cases} x^{\gamma}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} & for\ 0 \le x < \frac{1}{a(1-\alpha)}, a>0, \gamma>0, \alpha<1 \\ 0, & otherwise \end{cases} \tag{3.7}$$

$$f_2(y) = \begin{cases} e^{-y^{\rho}} & for\ 0 \le y < \infty, \rho>0 \\ 0, & otherwise \end{cases} \tag{3.8}$$

From (2.5) we have

$$I_{1\alpha} = \int_0^\infty \frac{1}{x} f_1(x) f_2\left(\frac{y}{x}\right) dx$$

$$= \int_0^d x^{\gamma-1}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-u^{\rho}x^{-\rho}} dx \quad where\ d=\frac{1}{a(1-\alpha)}$$

$$= \int_0^\infty x^{\gamma-1}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-bx^{-\rho}} dx \quad where\ b=u^{\rho}, \alpha<1 \tag{3.9}$$

The Mellin transform of $f_3 = I_{2\alpha}^{(d)}$ is the product of the Mellin transforms of f_1 and f_2

$$M_{f_3}(s) = M_{f_1}(s) M_{f_2}(s)$$

$$M_{f_1}(s) = \int_0^{\frac{1}{a(1-\alpha)}} x^{\gamma+s-1}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} dx$$

Putting $a(1-\alpha)x = t$, we get

$$M_{f_1}(s) = \frac{1}{[a(1-\alpha)]^{\gamma+s}} \int_0^1 t^{\gamma+s-1}(1-t)^{\frac{1}{1-\alpha}} dt$$

$$= \frac{1}{[a(1-\alpha)]^{\gamma+s}} \frac{\Gamma(\gamma+s)\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\Gamma\left(1+\gamma+\frac{1}{1-\alpha}+s\right)}, \mathrm{Re}(\gamma+s)>0, \alpha<1 \tag{3.10}$$

Where $\mathrm{Re}(\gamma+s)>0, \mathrm{Re}\left(\frac{1}{1-\alpha}+\gamma+s\right)>0, \alpha<1.$

From equation (3.5) we get

$$M_{f_2}(s)=\frac{1}{\rho}\Gamma\left(\frac{s}{\rho}\right), \quad \mathrm{Re}(s)>0 \tag{3.11}$$

From (3.10) and (3.11)

$$M_{f_3}(s)=\frac{1}{\rho[a(1-\alpha)]^{\gamma+s}}\frac{\Gamma(\gamma+s)\Gamma\left(\frac{1}{1-\alpha}+1\right)\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(1+\gamma+\frac{1}{1-\alpha}+s\right)}$$

Then the density of u denoted by $f_3(u)$ is available from the inverse Mellin transform.

$$f_3(u)=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\rho[a(1-\alpha)]^{\gamma}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\gamma+s)\Gamma\left(\frac{s}{\rho}\right)}{\Gamma\left(1+\frac{1}{1-\alpha}+\gamma+s\right)}[a(1-\alpha)u]^{-s}ds \tag{3.12}$$

Comparing (3.9) and (3.13)

$$\int_0^d x^{\gamma-1}[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}}e^{-bx^{-\rho}}dx$$

$$=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\rho[a(1-\alpha)]^{\gamma}}\overline{H}_{1,2}^{2,0}\left[a(1-\alpha)b^{\frac{1}{\rho}}\middle|\begin{matrix}\left(1+\frac{1}{1-\alpha}+\gamma,1\right)\\(\gamma,1),\left(0,\frac{1}{\rho}\right)\end{matrix}\right], \alpha<1$$

Therefore,

$$I_{2\alpha}^{(d)}=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\rho[a(1-\alpha)]^{\gamma}}\overline{H}_{1,2}^{2,0}\left[a(1-\alpha)b^{\frac{1}{\rho}}\middle|\begin{matrix}\left(1+\frac{1}{\alpha-1}+\gamma,1\right)\\(\gamma,1),\left(0,\frac{1}{\rho}\right)\end{matrix}\right]$$

Where $a>0, b>0, \gamma>0, \rho>0, \alpha<1, \ \alpha\to 1$.

Special Cases

If $\frac{1}{\rho}$ is an integer then put $\frac{1}{\rho}=m$. Then following through the same procedure as before one has

$$I_{2\alpha}^{(d)}=\frac{\sqrt{m}(2\pi)^{\frac{1-m}{2}}\Gamma\left(\frac{1}{1-\alpha}+1\right)}{[a(1-\alpha)]^{\gamma}}\overline{G}_{1,m+1}^{m+1,1}\left[\frac{a(1-\alpha)b^m}{m^m}\middle|\begin{matrix}\left(1+\frac{1}{\alpha-1}+\gamma\right)\\0,\frac{1}{m},\ldots,\frac{m-1}{m},\gamma\end{matrix}\right]$$

Where $a>0, b>0, \gamma>0, \alpha<1$.

For the thermonuclear function for non-resonant thermonuclear reactions with high energy cut-off $\gamma-1=\nu, a=1, \rho=\frac{1}{2}$, then we get,

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$$I_{2\alpha}^{(d)}=\frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\sqrt{\pi}[a(1-\alpha)]^{\nu+1}}\bar{G}_{1,3}^{3,0}\left[\frac{(1-\alpha)b^2}{4}\middle|\begin{matrix}\left(\nu+\frac{1}{1-\alpha}+2\right)\\0,\frac{1}{2},\nu+1\end{matrix}\right]$$

Where $b>0,\nu>0,\alpha<1$. The importance of the above result is that $I_{2\alpha}^{(d)}$ gives an extension of the integral $I_2^{(d)}$ to a wider class of integrals through the pathway parameter α, and their solutions.

CONCLUSION

In the field of stellar, cosmological, and controlled fusion, for example, the core of the Sun is considered as the gravitationally stabilized solar fusion reactor. The probability for a thermonuclear reaction to occur in the solar fusion plasma depends mainly on two factors. One of them is the velocity distribution of the particles in the plasma and is usually given by the Maxwell-Boltmann distribution of Boltmann-Gibbs statistical mechanics. The other factor is the particle reaction cross-section that contains the dominating quantum mechanical tunneling probability through a Coulomb barrier, called Gamow factor. Particle reactions in the hot solar fusion plasma will occur near energies where the product of velocity distribution is a maximum. The product of velocity distribution function and penetration factor is producing the Gamow peak. Mathematically, the Gamow peak is a thermonuclear function. In case of taking into consideration electron screening of reactions in the hot fusion plasma, the Coulomb potential may change to a Yukawa-like potential. Taking into account correlations and long-range forces in the plasma, the Maxwell-Boltzmann distribution may show deviations covered by the distribution predicated by Tsallis statistics in terms of cut-off or depletion of the high-velocity tail of the distribution function. In this paper, closed-form representations have been derived for thermonuclear functions, thus for the Gamma peak, for Boltzmann-Gibbs and Tsallis statistics. For this purpose, generalized entropy of order α and the respective distribution function have been considered. The case $\alpha=1$ recovers the Maxwell-Boltzmann case. This general case is characterized by moving cut-off, respectively the upper integration limit of the thermonuclear function to infinity. The closed-form representations of thermonuclear functions are achieved by using generalized hypergeometric functions or $\bar{H}$ -functions and $\bar{G}$ -functions, respectively.

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