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Research Article

MATHEMATICAL RELATION BETWEEN CONJUGATE FOCAL DISTANCES AND OTHER VARIOUS PARAMETERS OF AN ELLIPSE

***Kalaimaran Ara**
Construction & Civil Maintenance Deptartment, Central Food Technological Research Institute (Council of Scientific & Industrial Research), Mysore-20, Karnataka, India
**Author for Correspondence*

ABSTRACT
Ellipse is one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant 'e' which is less than or equal to unity. According to the Keplar's law of Planetary Motion, the ellipse is very important in geometry and the field of Astronomy, since every planet is orbiting its star in an elliptical path and its star is as one of the foci. The focal distances is an important factor in study about planetary motion. The author has attempted to establish a new mathematical relation combining five properties related to focal distances of an ellipse together as a single. This may be very useful to the scholars for reference to higher level research works.

Key Words: *Ellipse, Conic Sections, Keplar's Law of Planetary Motion, Eccentricity of ellipse, Foci, Directrix, Semi-major axis, Semi-minor axis.*

INTRODUTION

An *ellipse* (Weisstein Eric, 2003) is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* (Weisstein Eric, 2003) is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. This property can be used, for instance, in medicine. A point inside the ellipse which is the midpoint of the *line segment* (Weisstein Eric, 2003) linking the two foci is called centre. The longest and shortest diameters of an ellipse is called *Major axis* (Weisstein Eric, 2003) and *Minor axis* (Weisstein Eric, 2003) respectively. The two points that define the ellipse is called foci. The *eccentricity* Christopher (Clapham and James Nicolson, 2009), of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A line segment linking any two points on an ellipse is called *chord* (Weisstein Eric, 2003). A straight line passing an ellipse and touching it at just one point is called *tangent* (Weisstein Eric W, 2003). A straight line that passing through the centre of ellipse is called *diameter* (Weisstein Eric, 2003). A straight line that passing through the centre of the parallel lines to the diameter ellipse is called *conjugate diameter* (Weisstein Eric, 2003). The distances between foci and a point is called *focal distances* (Weisstein Eric, 2003) of that point.

There are some existing properties of ellipse such as "sum of focal distances is a constant" (called focal constant), reflection property of focus of the ellipse, etc. Now an attempt has been made by the Author to develop mathematical theorems regarding focal distance of an ellipse and the various parameters such as semi diameters, semi conjugate diameters, semi major axis semi minor axis, tangent and normal. The new properties have been derived mathematically. In this article, step by step derivations have been presented. The geometrical properties, which have been defined in this research article is very useful for those doing research work or further study in the field of Astronomy, Conics and Euclidean geometry, since this is also one of the important properties of an ellipse. This may also be very important to scientists who work in the field of *Optics* (John Sinclair, 2003).

MATERIALS AND METHODS

First Property of an Ellipse

Suppose a tangent is drawn to an ellipse at point 'P', 'O' is centre of the ellipse, 'T' is the intersection point of tangent and x-axis (line of major axis), 'OT' is the x-intercept of the extended tangent and 'Q' and 'R' are the foci. Then the product of pair focal distances from point 'P' is equal to the product of the distances of the foci 'Q' and 'R' from the intersection point 'T' minus square of tangent length 'PT'. (Ref. fig. 1)

Corollary Theorem for First Property of an Ellipse

Similarly, a tangent is drawn to an ellipse at point 'P', 'O' is centre of the ellipse, 'S' is the intersection point of tangent and y-axis (line of minor axis), 'OS' is the y-intercept of the extended tangent and 'Q' and 'R' are the foci. Then the product of pair focal distances from point 'P' is equal to the square of the distance of any one of the foci from the intersection point 'S' ('SQ' and 'SR' are equidistance) minus square of tangent length 'PT' (Ref. fig. 1).

Derivation of Equations for First Property

Referring figure 1, point O is the centre of an ellipse, P is the point anywhere on ellipse, point N is the projection of P on line of major axis, the point M is the projection of P on line of minor axis. Therefore, $NP \perp OT$ and $MP \perp OS$. , TS is the tangent drawn at P, OT and OS are the *x-intercept* **(**Weisstein Eric , 2003) (c) (with line of major axis) and *y-intercept* (Weisstein Eric, 2003) (with the line of minor axis) of extended tangent drawn at P. NT is the sub tangent. 'Q' and 'R' are the foci of the ellipse. 'PQ' and 'PR' are the pair focal distances from point 'P'. 'a' and 'b' are the semi-major axis and semi-minor axis of the ellipse respectively.

In right-angled triangle ONP, $\angle ONP = 90°$. Let, $\angle NTP = \theta°$.

We know the property, $ON \times OT = a^2$ Bali N.P (2005) (a)

$$\therefore ON = \frac{a^2}{OT} \qquad \text{------------------------[1.1]}$$

In fig. 1, $NT = OT - ON$

Substituting eqn. 1.1 in above

$$NT = OT - \frac{a^2}{OT}$$

$$\therefore NT = \frac{OT^2 - a^2}{OT} \qquad \text{---------------------------[1.2]}$$

We know the property, $OM \times OS = b^2$ Bali N.P (2005) (b)

Substituting, $OM = NP$ in above eqn. [Referring fig.1, $OM = NP$]

$$\therefore NP \times OS = b^2 \qquad \text{------------------------------[1.3]}$$

In right angled triangle TOS, $\tan\theta° = \dfrac{OS}{OT}$

$\therefore OS = OT \times \tan\theta°$ and substituting in eqn. 1.3

$$NP \times (OT \times \tan\theta°) = b^2$$

$$\therefore NP = \frac{b^2}{OT \times \tan\theta} \qquad \text{------------------------------[1.4]}$$

(i) Determination of 'OT'

In right angled triangle TNP,

$$\tan\theta = \frac{NP}{NT}$$

Substituting, eqn. 1.4 and 1.2 in above eqn., we get

$$= \left(\frac{b^2}{OT \times \tan\theta}\right) \times \left(\frac{OT}{OT^2 - a^2}\right)$$

$$\tan\theta = \frac{b^2}{\tan\theta\,(OT^2 - a^2)}$$

$$\therefore \tan^2\theta = \frac{b^2}{OT^2 - a^2}$$

$$\therefore OT^2 - a^2 = \frac{b^2}{\tan^2\theta}$$

$$\therefore OT^2 = \frac{b^2}{\tan^2\theta} + a^2$$

$$\therefore OT^2 = \frac{b^2 + a^2\tan^2\theta}{\tan^2\theta}$$

Substituting $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$OT^2 = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{\sin^2\theta}$$

$$OT = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta} \qquad \text{-------------------------} \; [1.5]$$

(ii) Determination of 'QN'

In fig.1, $QN = ON - OQ$

Substituting, eqn.1.1 and $OQ = \sqrt{a^2 - b^2}$ in above

$$QN = \frac{a^2}{OT} - \sqrt{a^2 - b^2}$$

Substituting, eqn.1.5 in above

$$QN = \frac{a^2\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} - \sqrt{a^2 - b^2}$$

$$\therefore QN = \frac{a^2\sin\theta - \left(\sqrt{a^2 - b^2}\right)\left(\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\right)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad \text{-----------------} \; [1.6]$$

(iii) Determination of 'NP'

In right angled trianle PNT, $\tan\theta = \dfrac{NP}{NT}$

$\therefore NP = NT \times \tan\theta$

Substituting eqn. 1.4 in above eqn. we get

$\therefore NP = NT \times \tan\theta$

Referring eqn. 1.4, $PN = \dfrac{b^2}{OT \times \tan\theta}$

Substituting eqn. 1.5 in above, we get

$$NP = \frac{b^2}{\left(\dfrac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta}\right) \times \tan\theta}$$

Put, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and simplifying, we get

$$NP = \frac{b^2\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad \text{----------------------} \; [1.7]$$

(iv) Determination of 'PQ' and 'PR'

In right-angled triangle PNQ,

$PQ^2 = NP^2 + QN^2$

Substituting eqn.1.7 and 1.6 in above eqn. we get

$$PQ^2 = \frac{b^4\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta} + \frac{\left[a^2\sin\theta - \left(\sqrt{a^2-b^2} \times \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\right)\right]^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{b^4\cos^2\theta + a^4\sin^2\theta + a^4\sin^2\theta + a^2b^2\cos^2\theta - a^2b^2\sin^2\theta - b^4\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{\left(a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - a\sqrt{a^2\sin^2\theta - b^2\sin^2\theta}\right)^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\therefore PQ = \frac{a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - a\sqrt{a^2\sin^2\theta - b^2\sin^2\theta}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= a - a\sqrt{\frac{a^2\sin^2\theta - b^2\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\therefore PQ = a\left(1 - \sqrt{\frac{a^2\sin^2\theta - b^2\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \qquad \text{------------------------ [1.8]}$$

Similarly,

$$PR = a\left(1 + \sqrt{\frac{a^2\sin^2\theta - b^2\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \qquad \text{------------------------[1.9]}$$

(v) Product of ‘PQ’ and ‘PR’

$$(PQ) \times (PR) = a\left(1 - \sqrt{\frac{a^2\sin^2\theta - b^2\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \times a\left(1 + \sqrt{\frac{a^2\sin^2\theta - b^2\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}}\right)$$

$$= \frac{a^2(b^2\sin^2\theta + b^2\cos^2\theta)}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$(PQ) \times (PR) = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \qquad \text{--------------------- [1.10]}$$

(Vi) Determination of ‘TQ’ and ‘TR’

Referring fig. 1, $TQ = OT - OQ$

$\therefore TQ = OT - \sqrt{a^2 - b^2}$

Substituting, eqn. 1.5 in above eqn., we get

$$TQ = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta} - \sqrt{a^2 - b^2}$$

$$\therefore TQ = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - \sin\theta\sqrt{a^2 - b^2}}{\sin\theta} \qquad \text{----------------- [1.11]}$$

Similarly,

$$TR = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + \sqrt{a^2\sin^2\theta - b^2\sin^2\theta}}{\sin\theta} \qquad \text{---------------[1.12]}$$

(vii) Product of ‘TQ’ and ‘TR’

$$(TQ)\times(TR) = \left(\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - \sin\theta\sqrt{a^2-b^2}}{\sin\theta}\right)\times\left(\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + \sin\theta\sqrt{a^2-b^2}}{\sin\theta}\right)$$

$$= \frac{a^2\sin^2\theta + b^2\cos^2\theta - a^2\sin^2\theta + b^2\sin^2\theta}{\sin^2\theta}$$

$$= \frac{b^2(\cos^2\theta+\sin^2\theta)}{\sin^2\theta}$$

$$\therefore (TQ)\times(TR) = \frac{b^2}{\sin^2\theta} \qquad \text{------------------------}[1.13]$$

(viii) Determination of 'NT'
Referring eqn. 1.2

$$NT^2 = \left(\frac{OT^2 - a^2}{OT}\right)^2$$

$$\therefore NT^2 = \left[\frac{\left(\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta}\right)^2 - a^2}{\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta}}\right]^2$$

$$= \left[\frac{a^2\sin^2\theta + b^2\cos^2\theta - a^2\sin^2\theta}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right]^2$$

$$= \left[\frac{b^2\cos^2\theta}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right]^2$$

$$\therefore NT^2 = \frac{b^4\cos^4\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)} \qquad \text{---------------------}[1.14]$$

(ix) Determination of 'PT'

$$PT^2 = PN^2 + NT^2$$

Substituting, eqn. 1.7 and 1.12 in above eqn., we get

$$PT^2 = \frac{b^4\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta} + \frac{b^4\cos^4\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}$$

$$\therefore PT^2 = \frac{\left(b^4\cos^2\theta\sin^2\theta\right) + b^4\cos^4\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}$$

$$\therefore PT^2 = \frac{b^4\cos^2\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}$$

$$\therefore PT = \sqrt{\frac{b^4\cos^2\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}} \qquad \text{---------------------}[1.15]$$

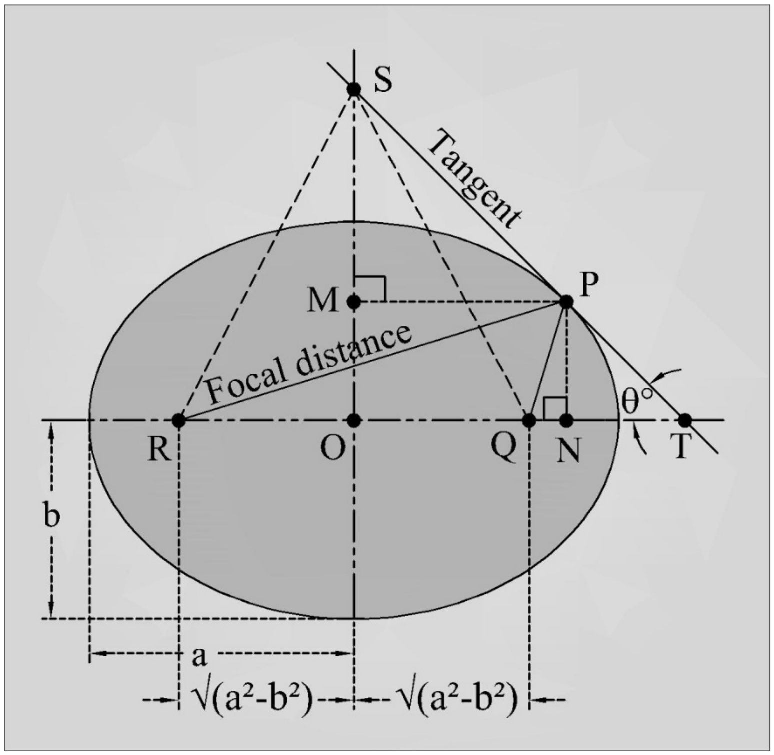

Figure 1: An Ellipse with tangent and Focal distances at point P

(x) Final proof of the theorem
Adding Eqn. 1.15 and 1.10, we get

$$PT^2 + (PQ \times PR) = \left(\frac{b^4\cos^2\theta}{\sin^2\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right)}\right) + \left(\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\right)$$

$$\therefore PT^2 + (PQ \times PR) = \frac{b^2}{\sin^2\theta} \qquad ---------------------- [1.16]$$

Equating eqns. 1.13 & 1.16

$$PT^2 + (PQ \times PR) = (TQ) \times (TR)$$

$$\therefore (PQ \times PR) = (TQ) \times (TR) - PT^2 \qquad ----------------------[1.17]$$

Similarly,

$$\therefore (PQ \times PR) = (SQ) \times (SR) - PS^2$$

In any ellipse the SQ and SR are the equidistance from S due to symmetry about OS.

Therefore, this can be rewritten as $(PQ \times PR) = SQ^2 - PS^2$ ------------[1.18]

The eqn. 1.17 and 1.18 are the mathematical expression for one of the properties of an ellipse

RESULTS AND DISCUSSION

Circle is a particular case of an ellipse.
Put, $PQ = PR = r$ and $TQ = TR = OT$ in eqn [1.17]
$PT^2 + r^2 = OT^2$.
This is very popular property of a tangent to a Circle

2. Second property of an ellipse

Suppose a tangent is drawn to an ellipse at point ‘P’, ‘O’ is centre of the ellipse, 'T' and 'S' are the intersection points of tangent with x-axis (line of major axis) and x-axis (line of major axis) respectively, 'OT' is the x-intercept and 'OS' is the y-intercept of the extended tangent and ‘Q’ & ‘R’ are the foci. Then the product of distances of the intersection points of extended tangent from 'P' (‘PT’ & 'PS') (Ref. fig. 1).

Derivation of equations for second property

Referring figure 1, point O is the centre of an ellipse, P is the point anywhere on ellipse, point N is the projection of P on line of major axis, point M is the projection of P on line of minor axis. Therefore, NP $\perp$ OT and MP $\perp$ OS. PT is the extended tangent to the line of major axis, PS is the extended tangent to the line of minor axis, PT is the tangent. Q and R are the foci of the ellipse. PQ and PR are the pair focal distances at P. In right-angled triangle ONP, $\angle$ONP = 90°. Let, $\angle$NTP =θ°.

(xi) Determination of ‘TS’

In right angled triangle TOS,

$$TS^2 = OT^2 + OS^2$$

$$TS = \sqrt{OT^2 + OS^2} \qquad \text{---------------------------} [2.1]$$

In right angled triangle TOS,

$$\frac{OS}{OT} = \tan\theta$$

OS = OT tanθ and substituting in equation [2.1], we get

$$TS = \sqrt{OT^2 + OT^2\tan^2\theta}$$

$$= OT\sqrt{1 + \tan^2\theta}$$

$$= OT\sqrt{\frac{1}{\cos^2\theta}}$$

$$\therefore TS = \frac{OT}{\cos\theta}$$

Substituting equation [1.5] in above, we get

$$TS = \left(\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta}\right) \times \left(\frac{1}{\cos\theta}\right)$$

$$\therefore TS = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta\,\cos\theta} \qquad \text{--------------------------} [2.2]$$

(xii) Determination of ‘PS’

$$PS = TS - PT$$

Substituting equation [1.14] and [1.2] in above equation, we get

$$= \left(\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\sin\theta\,\cos\theta}\right) - \left(\frac{b^2\cos\theta}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right)$$

$$PS = \frac{a^2\sin^2\theta + b^2\cos\theta - b^2\cos^2\theta}{\sin\theta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= \frac{a^2\sin^2\theta}{\sin\theta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$PS = \frac{a^2 \sin\theta}{\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \quad ---------------------- [2.3]$$

(xiii) Product of 'PT' and 'PS'

Eqn. [1.15] $\Longrightarrow \quad PT = \dfrac{b^2\cos\theta}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Eqn. [2.3] $\Longrightarrow \quad PS = \dfrac{a^2\sin\theta}{\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

$$\therefore PT \times PS = \left(\frac{b^2\cos\theta}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \times \left(\frac{a^2\sin\theta}{\cos\theta\sqrt{a^2\sin^2 a + b^2\cos^2\theta}}\right)$$

$$\therefore PT \times PS = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \quad ---------------------- [2.4]$$

(xiv) Final proof of the theorem

Eqn. 1.10 $\Rightarrow PQ \times PR = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

Eqn. 2.4 $\Rightarrow PT \times PS = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

Equating, eqn. [1.10] & [2.4]

$PQ \times PR = PT \times PS$ ---------------------------- [2.5]

The eqn. 2.5 is the mathematical expression for one of the properties of an ellipse

3. Third property of an ellipse

Suppose a tangent is drawn to an ellipse at point 'P', points 'J' and 'K' are the intersection point of extended normal with x-axis (line of major axis) and y-axis (line of minor axis) respectively and points 'Q' and 'R' are the foci of the ellipse. Then the product of pair focal distances 'PQ' & 'PR' of point 'P' anywhere on ellipse is equal to the product of the distances of the intersection point of extended normal from the point 'P' ('PJ' and 'PK') (fig. 2).

Derivation of equations for third property

Referring figure 2, point O is the centre of an ellipse, P is the point anywhere on ellipse, point N is the projection of P on line of major axis, point M is the projection of P on line of minor axis. Therefore, NP ⊥ OT and MP ⊥ OS. PT is the extended tangent to the line of major axis, PS is the extended tangent to the line of minor axis, PT is the tangent and PJ is the normal. NT is the sub tangent and NJ is sub normal. Q and R are the foci of the ellipse. PQ and PR are the pair focal distances at P. In right-angled triangle ONP, ∠ONP = 90°. Let, ∠NTP = θ°, ∠NOP = β°

(xv) Determination of PJ

Referring figure. 2. In right- angled triangle JPT, ∠JPT = 90°. Let ∠PTJ = θ°.

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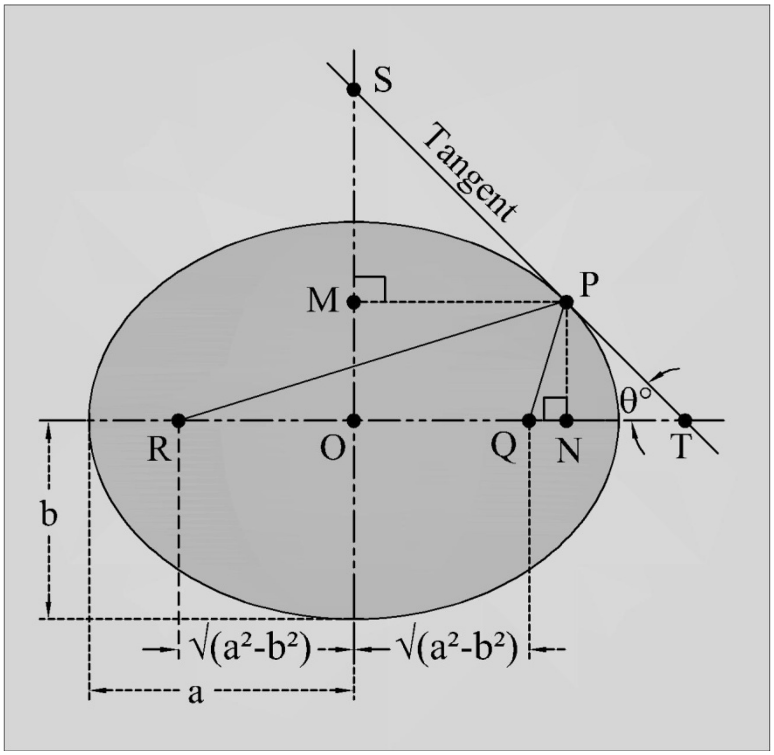

Figure 2: An Ellipse with tangent and normal at point P

$\tan\theta° = \dfrac{PJ}{PT}$

Therefore, $PJ = PT \times \tan\theta°$

Substituting eqn. [1.15] in above eqn., we get

$$PJ = \left(\frac{b^2\cos\theta°}{\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \times \left(\frac{\sin\theta°}{\cos\theta°}\right)$$

$$PJ = \frac{b^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad \text{--------------------------} [3.1]$$

(xvi) Determination of PK

Referring fig. 1. In right- angled triangle JPS, ∠JPS = 90°.

If ∠PTJ = θ°, ∠SKP is also equal to θ°

$\tan\theta° = \dfrac{PS}{PK}$

Therefore, $PK = \dfrac{PS}{\tan\theta°}$

Substituting eqn. [2.3] in above eqn., we get

$$PK = \left(\frac{a^2\sin\theta°}{\cos\theta°\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}\right) \times \left(\frac{\cos\theta}{\sin\theta°}\right)$$

$$PK = \frac{a^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad \text{--------------------------} [3.2]$$

(xvii) Product of PJ & PK

Eqn. [3.1] $\Rightarrow$ $$PJ = \frac{b^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

Eqn. [3.2] $\Rightarrow$ $$PK = \frac{a^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\therefore PJ \times PK = \left(\frac{b^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right) \times \left(\frac{a^2}{\sqrt{a^2\sin^2 a + b^2\cos^2\theta}}\right)$$

Therefore, $$PJ \times PK = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \quad \text{------------------} [3.3]$$

(xviii) Final proof of the theorem

Eqn. 1.10 $\Rightarrow$ $$PQ \times PR = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

Eqn. 3.3 $\Rightarrow$ $$PJ \times PK = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

Equating, eqns. [1.10] & [3.3]

$$PQ \times PR = PJ \times PK \quad \text{------------------} [3.4]$$

The eqn. [3.4] is the mathematical expression for one of the properties of ellipse.

4. Fourth property of an ellipse

Suppose a tangent is drawn to an ellipse at point ‘P’, point 'O' is the centre of the ellipse, point ‘U’ is the antipodal point Weisstein (Eric W, 2012) to the 'P' (therefore 'PU' is the diameter through 'P'), 'OL' is the semi conjugate diameter to diameter 'PU' and points ‘Q’ and ‘R’ are the foci of the ellipse. Then the product of pair focal distances ‘PQ’ & ‘PR’ of point 'P' anywhere on ellipse is equal to the product of pair focal distances of point 'P' anywhere on ellipse is equal to the square of the semi conjugate diameter ‘OL’ drawn to the diameter 'PU' (Ref. fig. 3).

Derivation of equations for fourth property

Referring figure 3, point O is the centre of an ellipse, P is the point anywhere on ellipse, point N is the projection of P on line of major axis, point M is the projection of P on line of minor axis. Therefore, NP $\perp$ OT and MP $\perp$ OS. PQ and PR are the pair focal distances at P. In right-angled triangle ONP, $\angle$ONP = 90°. Let, $\angle$NTP = θ°, $\angle$NOP = β°. 'PU' is the diameter to the ellipse through 'P'. 'EL’ is the semi conjugate diameter to ‘PU’ through point ‘O’. 'OL' is the semi conjugate diameter to the ellipse to diameter 'PU' through 'P'.

(xix) Determination of ‘OL’

In fig.1, The ellipse is symmetry with respect to both major axis and minor axis. Therefore, the semi diameter of the ellipse at angle θ° can be determined by using the following polar equation of ellipse [Daniel Zwillinger (2002)];

The *equation of ellipse with polar coordinates* (Weisstein Eric W, 2003) is

$$OL = \frac{ab}{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}$$

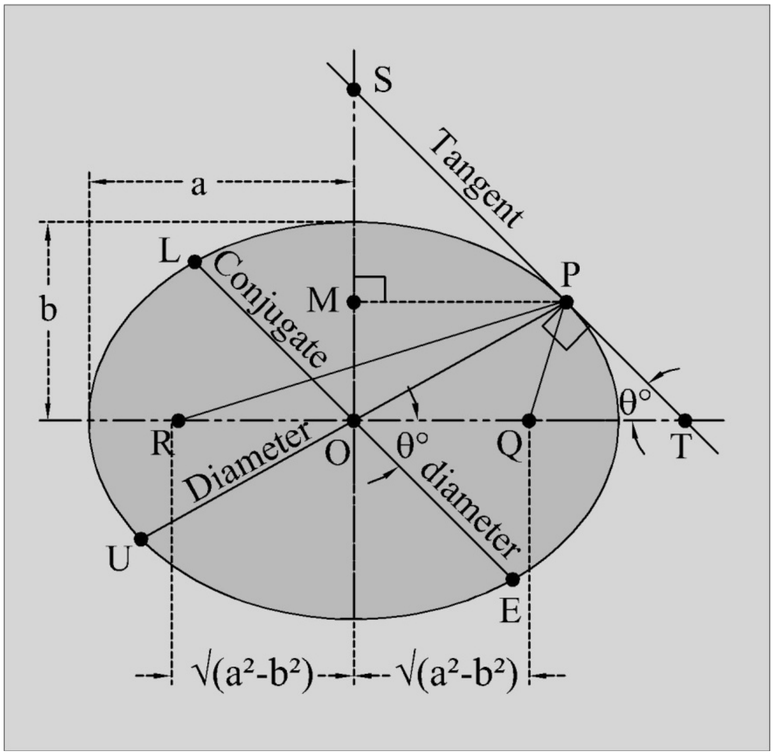

Figure 3: An Ellipse with tangent, diameter and conjugate diameter

$$\therefore \quad OL^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \qquad \text{------------------------ [4.1]}$$

(xx) Final proof of the theorem

Eqn. 1.10 $\Rightarrow PQ \times PR = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

Eqn. 4.1 $\Rightarrow OL^2 = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

Equating the eqns. [1.10] and [4.1]

$$PQ \times PR = OL^2 \qquad \text{----------------------------- [4.2]}$$

The eqn. [4.2] is the mathematical expression for one of the properties of ellipse.

5. Fifth property of an ellipse

Suppose a tangent is drawn to an ellipse at point ‘P’, point 'O' is the centre of the ellipse, OP is the semi diameter to the ellipse through 'P' and points ‘Q’ and ‘R’ are the foci.. Then the product of pair focal distances ‘PQ’ & ‘PR’ of point 'P' anywhere on ellipse is equal to the product of pair focal distances of point 'P' anywhere on ellipse is equal to the sum of square of the one of the side of the rhombus (Weisstein Eric W, 2003) formed by connecting all extremities of the ellipse (BC) minus square of the semi diameter ‘OP’ drawn through 'P' (Ref. fig. 4).

Derivation of equations for fifth property

Referring figure 4, point O is the centre of an ellipse, P is the point anywhere on ellipse, point N is the projection of P on line of major axis, point M is the projection of P on line of minor axis. Therefore, NP ⊥ OT and MP ⊥ OS. Q and R are the foci of the ellipse. Point U is the antipodal point to P, PU is the diameter through P therefore, OP is the semi diameter of the ellipse through P, A & C are the *extremities* (Bali, 2005) of ellipse about major axis and B & D are the extremities of the ellipse about minor axis. PQ and PR are the pair focal distances from P.

In right-angled triangle ONP, ∠ONP = 90°. Let, ∠NOP = β°. Q and R are the foci of the ellipse. In right-angled triangle ONP, ∠ONP = 90°. Let, ∠NOP = β°. We know that

$NP = OP \times \sin\beta°$ ——————————————————— [5.1]

$ON = OP \times \cos\beta°$ ——————————————————— [5.2]

$QN = ON - OQ$

We know already that $OQ = \sqrt{a^2 - b^2}$ ——————————————————— [5.3]

Substituting [5.2] and [5.4]

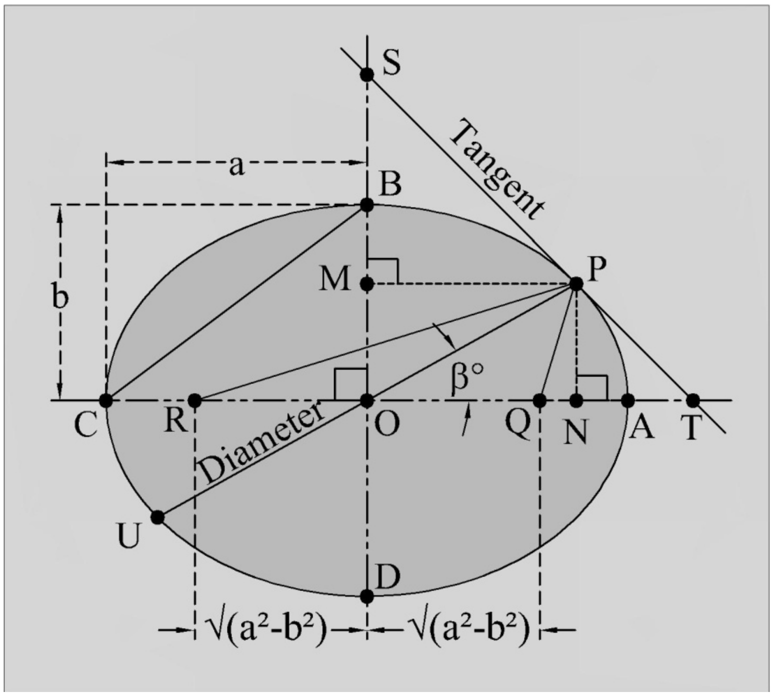

Figure 4: An Ellipse with tangent, diameter and focal distances

$\therefore QN = (OP \times \cos\beta°) - \sqrt{a^2 - b^2}$ ——————————————————— [5.4]

In right-angled triangle QNP, ∠QNP = 90°. Let, ∠NOP = β°. We know that

$PQ^2 = QN^2 + NP^2$

Substituting eqns. [5.4] & [5.1] in above eqn.

$$PQ^2 = \left(OP \times \cos\beta° - \sqrt{a^2 - b^2}\right)^2 + (OP \times \sin\beta°)^2$$

$$\therefore PQ^2 = \left(OP^2 \times \cos^2\beta°\right) + a^2 - b^2 - \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right) + \left(OP^2 \times \sin^2\beta°\right)$$

$$\therefore PQ^2 = \left[OP^2 \times (\cos^2\beta° + \sin^2\beta°)\right] + a^2 - b^2 - \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right) + OP^2$$

$$\therefore PQ^2 = OP^2 + a^2 - b^2 - \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right) \quad \text{———————— [5.5]}$$

In this fig. 1

$RQ = RO + OQ$

We know already that $RO = \sqrt{a^2 - b^2}$

$\therefore RQ = \sqrt{a^2 - b^2} + (OP \times \cos\beta°)$ ——————————————————— [5.6]

In right-angled triangle ONP, ∠ONP = 90°. Let, ∠NOP = β°. We know that

$PR^2 = RQ^2 + PQ^2$

Substituting eqns. [5.6] and [5.1] in above eqn.

$$PR^2 = \left[\sqrt{a^2 - b^2} + (OP \times \cos\beta°)\right]^2 + (OP \times \sin\beta°)^2$$

$$\therefore PR^2 = a^2 - b^2 + \left(OP^2 \times \cos^2\beta°\right) + \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right) + \left(OP^2 \times \sin^2\beta°\right)$$

$$\therefore PR^2 = \left\{OP^2 \times \left(\cos^2\beta° + \sin^2\beta°\right)\right\} + a^2 - b^2 + \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right)$$

$$\therefore PR^2 = OP^2 + a^2 - b^2 + \left(2\sqrt{a^2 - b^2} \times OP \times \cos\beta°\right) \quad \text{------------------[5.7]}$$

Adding [5.5] and [5.7],

$$\therefore PQ^2 + PR^2 = \left(2 \times OP^2\right) + 2a^2 - 2b^2 \quad \text{------ ------------- [5.8]}$$

$$(PQ + PR)^2 - 2(PQ)(PR) = \left(2 \times OP^2\right) + 2a^2 - 2b^2$$

We already know that $PQ + PR = 2a$ [Eric Weisstein (2002)] and substituting in above equation

Therefore, $4\square^2 - 2(PQ)(PR) = 2OP^2 + 2a^2 - 2b^2 \quad \text{------------------- [5.9]}$

$$\therefore -2(PQ)(PR) = 2OP^2 + 2a^2 - 2b^2 - 4a^2$$

$$\therefore -2(PQ)(PR) = 2OP^2 - 2b^2 - 2a^2$$

$$\therefore 2(PQ)(PR) = 2a^2 + 2b^2 - 2OP^2$$

$$\therefore 2(PQ)(PR) = 2\left(a^2 + b^2\right) - 2OP^2$$

$$\therefore PQ \times PR = a^2 + b^2 - OP^2 \quad \text{-------------------------- [5.10]}$$

In right triangle BOC, $\angle NOP$ is right angle and we know that OB = b and OC = a.

Therefore, $BC^2 = OB^2 + OC^2 = a^2 + b^2$ and substitute $a^2 + b^2 = BC^2$ in eqn. 5.10

Therefore, $\quad PQ \times PR = BC^2 - OP^2 \quad \text{---------------------[5.11]}$

Eqn. 5.11 is mathematical expression for one of the properties of ellipse.

Equating the eqns. [1.17], [1.18], [2.5], [3.4], [4.2] and [5.11] we get

$$PQ \times PR = (TQ \times TR) - PT^2 = SQ^2 - PS^2 = PT \times PS = PJ \times PK = BC^2 - OP^2 = OL^2 \quad \text{--- [5.12]}$$

where, a and b are semi-major axis and semi-minor axis respectively. Where, ‘PQ’ and ‘PR’ are the pair focal distances of the ellipse. The equation 5.12 is the mathematical expression of relation between conjugate focal distances and other various parameters of an ellipse.

RESULTS AND DISCUSSION

An example of ellipse is drawn in the AutoCAD to verify the theorem with the following parameters.

a = major axis = 4 units, b = minor axis = 3 units, a tangent is drawn at P with an angle θ° = 30°. The distances are measured from the AutoCAD drawing and these values are shown in the table-1.

Table 1: Various parameters of the example

Parameter	Mathematical term	Unit length
ON	Abscissa of P (x- coordinate)	2.4400
OM	Ordinate of P (y- coordinate)	2.3772
OT	x- intercept of tangent	6.5574
OS	y- intercept of tangent	3.7859
PQ	First focal distance of P	2.3861
PR	Second focal distance of P	5.6139
PT	-	4.7544

PS	-	2.8174
PJ	-	2.7450
PK	-	4.8800
TQ	-	3.9117
TR	-	9.2032
SQ , SR	-	4.6188
OP, OF	Semi diameter of ellipse through P	3.4066
OL , OE	Semi conjugate dia. of ellipse through P	3.6600
AB,BC, CD, AD	$\sqrt{a^2+b^2}$	5.0000

The above values have been substituted in the eqns. [1.17], [1.18], [2.5], [3.4], [4.2] and [5.11]

(i)	$PQ \times PR$	$= 2.3861 \times 5.6139$	$= 13.3953$	sq.units
(ii)	$(TQ \times TR) - PT^2$	$= (3.9117 \times 9.2032) - 4.7544^2$	$= 13.3958$	sq.units
(iii)	$SQ^2 - PS^2$	$= 4.6188^2 - 2.8174^2$	$= 13.3956$	sq.units
(iv)	$PT \times PS$	$= 4.7544 \times 2.8174$	$= 13.3951$	sq.units
(v)	$PJ \times PK$	$= 2.7450 \times 4.8800$	$= 13.3956$	sq.units
(vi)	$BC^2 - OP^2$	$= 5^2 - 3.4066^2$	$= 13.3951$	sq.units
(vii)	OL^2	$= 3.6600^2$	$= 13.3956$	sq.units

From (i) to (vii), we understand that all the values are equal and hence the theorem is proved with numerical values which were measured from the example problem.

CONCLUSIONS

As per Keplar's first law of planetary motion, all Planets including Comets and Asteroids, which are in space rotate in elliptical path as its star is one of the foci of that ellipse. The applications of the elliptical mirrors and lens is also very important. The geometrical property, which has been defined in this research article is very useful for those doing research or further study in the field of Astronomy, *Conics* Borowski and Borwein, (1991) and Euclidean geometry, since this is also one of the important properties of an ellipse. In this article, the mathematical relation between Conjugate focal distances and other various parameters of the ellipse has been defined very clearly with appropriate derivation for the equations and result. The author has derived necessary equations for geometrical properties of conjugate diameter drawn to diameter of an ellipse and developed a new geometrical theorems for it.

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