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ON DUALITY IN LINEAR PROGRAMMING UNDER VAGUE ENVIRONMENT

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ABSTRACT
Present Paper improves the model of Bector and Chandra, (2002) on duality in fuzzy linear programming by using vague environment. Numerical problem discussed by them has also been worked out through our model.

Key Words: *Linear Primal-Dual Problems, Fuzzy Environment, Tolerance, Membership Function, Non Membership Function*

INTRODUCTION

A number of researchers have exhibited their interest in the topic of fuzzy linear programming after (Zadeh, 1965) developed the concept of fuzzy set theory. Solution of Linear programming (LP) problems lead to an answer about the allocation of resources through the solution of the primal problem (Hadley, 1971). The solution to the dual problem does the market valuation of the resources. In the crisp situations, solution of primal (dual) implicitly provides the solution dual (primal). In fuzzy environment, instead of optimization, one works for the optimal satisfaction of aspiration levels for primal and dual both. The most basic result on duality in fuzzy linear programming are due to Rodder and Zimmerman, (1980); Hamacher *et al.,* (1978). In a generalization of maxmin and minmax problems in a fuzzy environment is present and thereby a pair of fuzzy dual linear programming problem is constructed. An economic interpretation of this duality in terms of market and industry is also included in 1980. The paper by Hamacher *et al.,* (1978) is mostly devoted to the study of sensitivity analysis in fuzzy linear programming. In FLP the optimization of primal-dual pair is important so as to reach as close as possible to the aspiration value in each case. Bector and Chandra, (2002) introduced a fuzzy pair of primal-dual problems modifying the construction of the fuzzy dual model of Rodder-Zimmermann formulation. Yang *et al.,* (1991) in their study of FLP found that the approach of dealing with non-linear membership functions is straightforward and computationally efficient. Present paper improves the Bector and Chandra model of Fuzzy primal-dual problem using by membership and non membership functions both.

Bector-Chandra Model for Fuzzy Dual

Let R^n denote the n-dimensional Euclidean space and R^n_+ be its non-negative part.

For $x, c \in R^n$ and the matrix A in $R^m \times R^n$ the Linear Primal(LP) and Linear Dual (LD) problems are expressed in the vector forms as following.

(LP) Maximize $c^T x$

subject to : $Ax \le b, \quad x \ge 0$

and

(LD) Minimize $b^T w$

subject to : $A^T w \ge c, \; w \ge 0$

Bector and Chandra, (2002) gave the fuzzy versions of LP and LD respectively in the sense of Zimmermann, (1980) as describes below:

Find $x \in R^n$ such that $c^T x \succ Z0; \quad Ax \prec b, \; x \ge 0$ (1)

and

Find $w \in R^m$ such that $b^T w \prec W0$, $A^T w \succ c$, $w \geq 0$ (2)

Here $\succ$ and^T $\prec$ are the fuzzy version of symbols $\geq$ and $\leq$ respectively, and have the linguistic interpretation as explained in (Zimmerman, 1991). Z0 and W0 respectively denote the aspiration levels of the two objectives $c^T x$, $b^T w$ .Further assuming p0 and p_i (i = 1, 2, 3, ….;m) to be subjectively chosen positive constants representing the admissible tolerance values associated with the objective function and m linear constraints of equation (1) respectively, the crisp equivalent of the fuzzy problem (1) in Bector and Chandra, (2002) is given as below:

Maximize λ

subject to $(\lambda - 1)p_0 \leq c^T x - Z_0$

$(\lambda - 1)\, p_i \leq b_i - A_i x$, i = 1,2,3…..,m (3)

$\lambda \leq 1$ and $x, \lambda \geq 0$

Where A_i and b_i are the i^{th} row of matrix A and i^{th} component of vector b (i=1,2,3,...,m) respectively. Similarly for q_0 and q_j (j=1,2,3,..,n) being the corresponding values for (2), the crisp equivalent for the problem (2) is given as below :

Minimize $(-\eta)$

subject to $(\eta - 1)q_0 = W_0 - b^T w$

$(\eta - 1)q_j \leq A_j^T w - c_j$, j = 1,2,3,….,n (4)

$\eta \leq 1$ and $w, \eta \geq 0$

Equation (3) to (4) is termed as fuzzy pair of primal-dual linear programming in Bector and chandra (2002).

Fuzzy model with piecewise linear membership functions

While a linear membership function permits an easy conversion of the FLP problem, into a crisp linear programming problem, yet in many cases membership functions are the best represented by non-linear functions. Yang *et al.*, (1991) have provided a representation of concave (and non-concave) non-linear membership functions that have been approximated by two and three linear segments.In which we use the following notations.

$\mu_0^p(x)$ / $\mu_0^D(w)$: Membership functions for Primal/Dual corresponding to objective function;

$\mu_{0k}^p(x)$ / $\mu_{0k}^D(w)$: Membership function for k^{th} (k = 1,2) linear segment of Primal/Dual corresponding to objective function;

$\mu_i^p(x)$ / $\mu_j^D(w)$: Membership functions for Primal/Dual to m/n constraints i=1,2,3,....,m and j=1,2,3...,n.

We start with the following linear membership function of Bector-Chandra model [1].

$$\mu_0^p(x) = \begin{cases} 1 & \text{if} \quad c^T x \geq Z_0 \\ 1 - \dfrac{Z_0 - c^T x}{p_0} & \text{if} \quad Z_0 - p_0 \leq c^T x < Z_0 \\ 0 & \text{if} \quad Z_0 - p_0 \geq c^T x \end{cases} \qquad (5)$$

$$\mu_i^p(x) = \begin{cases} 1 & if \quad A_i x \le b_i \\ 1 - \dfrac{A_i x - b_i}{p_i} & if \; b_i < A_i x \le b_i + p_i \\ 0 & if \quad A_i x > b_i + p_i \end{cases} \tag{6}$$

Let us now approximate $\mu_0^p(x)$ by two linear segment and define the membership functions corresponding to each segment as following:

$$\mu_{01}^p(x) = \begin{cases} 1 & if \quad c^T x \ge Z_{01} \\ 1 - \dfrac{Z_{01} - c^T x}{p_{01}} & if \quad Z_{01} - p_{01} \le c^T x < Z_{01} \\ 0 & if \quad c^T x < Z_{01} - p_{01} \end{cases} \tag{7}$$

and

$$\mu_{01}^p(x) = \begin{cases} 1 & if \quad c^T x \ge Z_{02} \\ 1 - \dfrac{Z_{02} - c^T x}{p_{02}} & if \quad Z_{02} - p_{02} \le c^T x < Z_{02} \\ 0 & if \quad c^T x < Z_{02} - p_{02} \end{cases} \tag{8}$$

Where $p_{01} + p_{02} > p_0$ and $Z_{01} + Z_{02} > Z_0$

Using $\lambda = \min\{\mu_{01}^p, \mu_{02}^p, \mu_1^p, \ldots\ldots, \mu_m^p\}$ [6](p245), the crisp equivalent of the fuzzy problem(1) becomes.

Maximize λ

subject to : $$\lambda \le 1 - \frac{Z_{01} - c^T X}{p_{01}} \tag{9}$$

$$\lambda \le 1 - \frac{Z_{02} - c^T X}{p_{02}} \tag{10}$$

$$\lambda \le 1 - \frac{A_i x - b_i}{p_i} \quad ,i=1,2,3,\ldots.m \tag{11}$$

$$\lambda \le 1 \text{ and } x, \lambda \ge 0 \tag{12}$$

Similarly, for the dual problem (2), membership functions are as follows.

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$$\mu_0^D(w)= \begin{cases} 1 & if \quad b^T w \leq w_0 \\ 1 + \dfrac{w_0 - b^T w}{q_0} & if\ w_0 < b^T w \leq w_0 + q_0 \\ 0 & if \quad w_0 + q_0 < b^T w \end{cases} \qquad (13)$$

$$\mu_j^D(w)= \begin{cases} 1 & if \quad A_j^T w \geq c_j \\ 1 + \dfrac{A_j^T w - c_j}{q_j} & if\ c_j - q_j \leq A_j^T w < c_j \\ 0 & if \quad c_j - q_j > A_j^T w \end{cases} \qquad (14)$$

Let us now approximate $\mu_0^D(w)$ by two linear segments and define the membership functions corresponding to each segment as follows.

$$\mu_{01}^D(w)= \begin{cases} 1 & if \quad b^T w \leq W_{01} \\ 1 + \dfrac{W_{01} - b^T w}{q_{01}} & if\ W_{01} < b^T w \leq W_{01} + q_{01} \\ 0 & if \quad b^T w > W_{01} + q_{01} \end{cases} \qquad (15)$$

$$\mu_{02}^D(w)= \begin{cases} 1 & if \quad b^T w \leq W_{02} \\ 1 + \dfrac{W_{02} - b^T w}{q_{02}} & if\ W_{02} < b^T w \leq W_{02} + q_{02} \\ 0 & if \quad b^T w > W_{02} + q_{02} \end{cases} \qquad (16)$$

Where $q_{01}+q_{02} > q_0$ and $W_{01}+W_{02} > W_0$

further letting $\eta =\{ \mu_{01}^D, \mu_{02}^D, \mu_1^D, \ldots\ldots \mu_n^D \}$ the crisp equivalent for the dual problem is .

Minimize($-\eta$)

subject to: $$\eta \leq 1+\frac{W_{01} - b^T w}{q_{01}} \qquad (17)$$

$$\eta \leq 1+\frac{W_{02} - b^T w}{q_{02}} \qquad (18)$$

$$\eta \leq 1+\frac{A_j^T w - c_j}{q_j}, \ j=1, 2, 3, \ldots.n \qquad (19)$$

$$\eta \leq 1, \text{ and } \eta, \ w \geq 0 \qquad (20)$$

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The equation (9) to (12) and (17) to (20) forms a pair is termed as fuzzy pair of primal-dual linear programming problem in our model.

Model for Linear Non-Membership Function

The Linear Non-Membership Function for Primal-dual pair is represented by the following notation.

ν_0^p (x) / ν_0^D (w): Non- Membership functions for Primal/Dual corresponding to objective function;

ν_{0k}^p (x)/ ν_{0k}^D (w) : Non- Membership function for kth (k=1,2) linear segment of Primal/Dual corresponding to objective function;

ν_i^p (x) / ν_j^D (w) : Non- Membership functions for Primal/Dual to m/n constraints i=1,2,3,....,m and j=1,2,3...,n, Let Z_0^* and W_0^* respectively denote the aspiration levels of the two objectives c^T x, b^T w and p_0^* and p_i^* are the positive constants representing the admissible tolerance values associated with the m linear constraints. Then the Linear Non-Membership functions is as follows.

$$\nu_0^p(x)= \begin{cases} 0 & if \quad c^T x \geq Z_0 \\ \dfrac{Z_0 - c^T x}{p_0^*} & if\ Z_0 - p_0^* \leq c^T x < Z_0 \\ 1 & if\ Z_0 - p_0^* > c^T x \end{cases} \tag{21}$$

$$\nu_i^p(x)= \begin{cases} 0 & if \quad A_i x \leq b_i \\ \dfrac{A_i x - b_i}{p_i^*} & if\ b_i < A_i x \leq b_i + p_{i\,0}^* \\ 1 & if\ A_i x > b_i + p_i^* \end{cases} \tag{22}$$

Let us now approximate ν_0^p (x) by two linear segments and define the non membership functions corresponding to each segment as following:

$$\nu_{01}^p(x)= \begin{cases} 0 & if \quad c^T x \geq Z_{01} \\ \dfrac{Z_{01} - c^T x}{p_{01}^*} & if \quad Z_{01} - p_{01}^* \leq c^T x < Z_{01} \\ 1 & if \quad c^T x < Z_{01} - p_{01}^* \end{cases} \tag{23}$$

$$\nu_{02}^p(x)= \begin{cases} 0 & if \quad c^T x \geq Z_{02} \\ \dfrac{Z_{02} - c^T x}{p_{02}^*} & if \quad Z_{02} - p_{02}^* \leq c^T x < Z_{02} \\ 1 & if \quad c^T x < Z_{02} - p_{02}^* \end{cases} \tag{24}$$

Using $\lambda_1 = \min\{ \nu_{01}^p, \nu_{02}^p, \nu_1^p, \ldots.., \nu_m^p \}$, the crisp equivalent of fuzzy problem becomes.

Maximize λ_1

Subject to: $\lambda_1 \le \frac{Z_{01} - c^T x}{p^*_{01}}$ (25)

$\lambda_1 \le \frac{Z_{02} - c^T x}{p^*_{02}}$ (26)

$\lambda_1 \le \frac{A_i x - b_i}{p^*_i}$, i=1,2,3,……,m (27)

$\lambda_1 \le 1$ and x, $\lambda_1 \ge 0$ (28)

Similarly non-membership function for dual problem

$$\nu_0^D(w) = \begin{cases} 0 \quad if & b^T w \le w_0^* \\ \frac{b^T w - W_0^*}{q_0^*} & if\ w_0^* < b^T w \le w_0^* + q_0^* \\ 1 \quad if & w_0^* + q_0^* < b^T w \end{cases} \tag{29}$$

$$\nu_j^D(w) = \begin{cases} 0 \quad if & A_j^T w \ge c_j \\ \frac{c_j - A_j^T w}{q_j^*} & if\ c_j - q_j^* \le A_j^T w < c_j \\ 1 \quad if & c_j - q_j^* > A_j^T w \end{cases} \tag{30}$$

Let us now approximate ν_0^p (x) by two linear segments and define the non membership functions corresponding to each segment as following:

$$\nu_{01}^D(w) = \begin{cases} 0 \quad if & b^T w \le W_{01}^* \\ \frac{b^T w - W_{01}^*}{q_{01}^*} & if\ W_{01}^* < b^T w \le W_{01}^* + q_{01}^* \\ 1 \quad if & b^T w > W_{01}^* + q_{01}^* \end{cases} \tag{31}$$

$$\nu_{02}^D(w) = \begin{cases} 0 \quad if & b^T w \le W_{02}^* \\ \frac{b^T w - W_{02}^*}{q_{02}^*} & if\ W_{02}^* < b^T w \le W_{02}^* + q_{02}^* \\ 1 \quad if & b^T w > W_{02}^* + q_{02}^* \end{cases} \tag{32}$$

Using η_1 = min{ $\nu_{01}^p, \nu_{02}^p, \nu_1^p, \ldots, \nu_m^p$ }, the crisp equivalent of fuzzy problem becomes.

Minimize $(-\eta_1)$

subject to : $\eta_1 \le \frac{c_j - A_j^T w}{q_j^*}$ (33)

$\eta_1 \le \frac{b^T w - W_{01}^*}{q_{01}^*}$ (34)

$\eta_1 \le \frac{b^T w - W_{02}^*}{q_{02}^*}$, j=1,2,3,….n, (35)

$\eta_1 \le 1$ and w, $\eta_1 \ge 0$ (36)

RESULTS AND DISCUSSION

Some Weak Duality Result

Standard Weak Duality Theorem:

Let x and w be any feasible solutions to problems (LP) and (LD) respectively, then $c^T x \le b^T w$ (37)

Modified Weak Duality Theorem:

Let(x, λ) be feasible for (3) and (w, η) be feasible for (5)

then $(\lambda - 1)p^T w + (\eta - 1)q^T x \le (b^T w - c^T x)$, (38)

where $p^T = (p_1, p_2, p_3, \ldots, p_m)$ and $q^T = (q_1, q_2, \ldots, q_n)$.

The modified weak duality theorem has been proved by Bector and Chandra, (2002) for the fuzzy environment where they consider membership functions to be linear. Obviously (38) reduce to (37) for $\lambda = \eta = 1$.Further since (38) does not involve tolerance values (p0 and q0) associated with the objective functions as suggested in (7),(8), (15)and (16), would not affect the result. This justifies the following remark.

Remark 1: The modified weak duality theorem is valid even when (x, λ) and (w, η) be feasible solutions for (9) to (12) and (17) to (20) respectively.

The inequality relating the relative difference of aspiration level Z_0 of $c^T x$ and W_0 of $b^T w$ in term p_0 and q_0 respectively, can be modified in the form of following result.

Theorem1: Let (x, λ) be the feasible solution for non-linear primal (9) to (12) and (w, η) be feasible Solution for non -linear dual (17) to (20) then

$(\lambda - 1)(p_{01} + p_{02}) + (\eta - 1)(q_{01} + q_{02}) \le 2(c^T x - b^T w) + \{(W_{01} + W_{02}) - (Z_{01} + Z_{02})\}$ (39)

Proof: From the equation (9) to (12) and (17) to(20) we have

$(\lambda - 1)(p_{01} + p_{02}) \le 2c^T x - (Z_{01} + Z_{02})$, (40)

$(\eta - 1)(q_{01} + q_{02}) \le (W_{01} + W_{02}) - 2b^T w$ (41)

After adding the above two inequalities we get the required result.

Theorem 2: Let (x^*, λ^*) and (w^*, η^*) be feasible solutions of (9) to (12) and (17) to (20) respectively and satisfying the following condition.

$(\lambda^* - 1)p^T w^* + (\eta^* - 1)q^T x^* = (b^T w^* - c^T x^*)$, (42)

$(\lambda^* - 1)(p_{01} + p_{02}) + (\eta^* - 1)(q_{01} + q_{02}) = 2(c^T x^* - b^T w^*) + \{(W_{01} + W_{02}) - (Z_{01} + Z_{02})\}$ (43)

$(Z_{01} + Z_{02}) \le W_{01} + W_{02}$ (44)

then(x^*, λ^*)is optimal to (9) to (12) and (w^*, η^*) is optimal to (17) to (20).

Proof: Let(x, λ) and (w, η) be some feasible solution of (9) to (12) and (17) to (20) respectively. Then by remarks 1,we have

$(\lambda-1)p^T w+(\eta-1)q^T x \le (b^T w-c^T x),$ (45)

from above equation and hypothesis of theorem 2 we have

$(\lambda-1)\, p^T w+(\eta-1)q^T x-(b^T w-c^T) \le (\lambda^*-1)\, p^T w^*+(\eta^*-1)\, q^T x^*-(b^T w^*-c^T x^*)$ (46)

From hypothesis (42) and equation (46) it can be implied that (x^*, λ^*, w^*, η^*) is optimal to the following problem whose maximum is zero.

Maximize $\{(\lambda-1)\, p^T w+(\eta-1)q^T x \le (b^T w-c^T x)\}$

Subject to :$(\lambda-1)\, p_{01} \le c^T x-Z_{01}$, (47)

$(\lambda-1)\, p_{02} \le c^T x-Z_{02}$, (48)

$(\eta-1)\, q_{01} \le W_{01}-b^T w$, (49)

$(\eta-1)\, q_{02} \le W_{02}-b^T w$, (50)

$(\lambda-1)\, p_i \le c_i - A_i x$, (51)

$(\eta-1)\, q_j \le A_j^T w - c_j$, (52)

$\lambda \le 1, \eta \le 1$, (53)

$x, w, \lambda, \eta, \ge 0$ (54)

Multiplying (42) by 2 and then adding to (43) ,we get

$2(\lambda^*-1)\, p^T w^* + 2(\eta^*-1)\, q^T x^* +(\lambda^*-1)\,(p_{01}+p_{02}) +(\eta^*-1)\,(q_{01}+q_{02}) + \{(Z_{01}+Z_{02})$
$- \quad (W_{01}+W_{02})\} = 0$

Since each term of the above expression is non-positive due to the hypothesis (44) and the fact that $\lambda^*, \eta^* \le 1$ it must therefore be separately equal to zero, Hence

$(\lambda^*-1)\,(p_{01}+p_{02}) = 0$ (55)

$(\eta^*-1)\,(q_{01}+q_{02}) = 0$ (56)

Further since, p_{01}, p_{02}, q_{01}, q_{02} >0 we also have

$(\lambda-1)\, p_{01} \le 0$, $(\lambda-1)\, p_{02} \le 0$ (57)

$(\eta-1)\, q_{01} \le 0$, $(\eta-1)\, q_{02} \le 0$ (58)

From (55),(56)and (57), (58), it is obvious that

$(\lambda-1)\, p_{01} \le (\lambda^*-1)\; p_{01}$, $(\lambda-1)\, p_{02} \le (\lambda^*-1)\; p_{02}$,

$(\eta-1)\, q_{01} \le (\eta^*-1)\; q_{01}$, $(\eta-1)\, q_{02} \le (\eta^*-1)\; q_{02}$,

Above inequalities imply that $\lambda \le \lambda^*$ and $-\eta \ge -\eta^*$. This proves the theorem.

Numerical Example

Consider the following primal-dual problem.

(P) Maximize 2 xs, subject to $x \le 1$, $x \ge 0$

(D) Minimize w, subject to w $\geq$ 2, w $\geq$ 0
firstly we solved the above problem by membership function
Taking Z_{01} = ½ and Z_{02} = 1 we get p_{01} = 3/2 and p_{02} = 4, p_1 = 2
The crisp equivalent of Fuzzy problem (P) becomes:
Maximize λ

subject to $\lambda \leq 1 - \frac{\frac{1}{2}-2x}{3/2}$ (59)

$\lambda \leq 1 - \frac{1-2x}{4}$ (60)

$2\lambda + x \leq 3$ (61)

$\lambda \leq 1$ and $x, \lambda \geq 0$ (62)

Using W_{01} = 1:70, W_{02} = 1, q_{01} = 0.30, q_{02} = 1.75 and q_1= 3 then crisp equivalent of fuzzy problem(D) can be obtained as following.
Minimize ($-\eta$)

subject to: $\eta \leq 1 + \frac{1.70-w}{.30}$, (63)

$\eta \leq 1 + \frac{1-w}{1.75}$, (64)

The optimal solutions (λ^*, η^*) of fuzzy primal-dual pair (P-D) are λ^*= 1 and η^* = .79 leading to x^* =1/2 and w^* = 1.37: the optimum pair of results (λ^*, η^*) for the same problem from Bector-Chandra model where λ^* = 1 and η^*= 3/4 leading to x^* = 1/2 and w^* = 5/4 It can be observed that the maximum value of λ has been fully achieved as Bector-Chandra model but minimum value of η in our model is (-.79) while the minimum value of in Bector-Chandra model is (-.75). Hence minimum value obtained from this model is better as compare to Bector -Chandra model also it also gives us the better value of w = 1.37 as compare to Bector -Chandra model value for w = 1.25
Similarly we solve above problem by non membership function.

CONCLUSIONS

A fuzzy primal -dual model has been worked out in this paper using linear membership functions and linear non membership function thereby improving a similar model given by Bector and Chandra, (2002). And from this method it is clear that fuzzy primal-dual model using membership function and non membership function is capable of better approximation. Some weak duality results in vague environment corresponding to linear membership function have also been established.

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