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Research Article

MATHEMATICAL STUDY OF COSMOLOGICAL MODELS WITH FRIEDMAN EQUATIONS

***R.K.Dubey[1] and Ajai Kumar Tripathi[2]**
[1]*Govt. P.G. Science College Rewa (M.P)*
[2]*Department of Mathematics Govt. P. G. College Shahdol (M.P.)*
**Author for Correspondence*

ABSTRACT
We have considered a cosmological model with Friedman equation for the expansion of universe. We obtain Friedman equation from first law of thermodynamics and Newtonian approach is also discussed. The evolution of the scale factor with time in a cosmological model in which the dominant contribution to the energy density arises from the radiation and matter within the universe is given .
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Key Words: *Friedman equation, Thermodynamics, Matter and Radiation –dominated universe.*

INTRODUCTION
The cosmological problem within the framework of general relativity consists of finding model of the physical universe which correctly predicts the result of astronomical observations and which is determined by those physical laws that describe the behavior of matter on scales up to those of clusters of galaxies .The simplest models of the expanding universe are those which are spatially homogeneous and isotropic at each instant of time.
On large scales the universe is homogeneous and isotropic, this idea is of such importance in cosmology that it has been elevated to the status of a Principle, and is usually known as the Cosmological Principle. It is useful to introduce a new variable related to the expansion parameter which is more directly observable. We call this variable the red shift z and we shall use it extensively from now on in describing the evolution of the universe because many of the relevant formulae are very simple when expressed in terms of this variable
We define the red shift of a luminous source, such as distant galaxy as

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

Properties of the universe can be discussed in terms of geometry, for this we must use general relatively to relate the geometry of space-time, expressed by the energy-momentum tensor $T_{\mu\upsilon}$. The Einstein equation is given as

$$R_{\mu\upsilon} - \frac{1}{2} R g_{\mu\upsilon} = \frac{8\pi G}{c^4} T_{\mu\upsilon} \qquad (1)$$

THE COSMOLOGICAL PRINCIPLE
The fundamental principal upon which most cosmological models are based is the cosmological principal, which is that the universe, at least on large scales, is homogeneous and isotropic. We may illustrate it in this way: All points in space ought to experience the same physical development, correlated in time in such a way that all points at a certain distance from an observer appear to beat the same stage of development . Or in the more concise way: On large spatial scales, the universe is homogeneous and isotropic. To study the cosmology, we have to believe that the place which we occupy in the universe is in no way special.
This leads to the definitions of *homogenous* and *isotropic*. Homogeneity is the statement that the universe looks the same at each point. We define that there is a cosmic time *t*, and *t* is constant in each of the space like slices, so each slice has no privileged points. Isotropy states that the universe looks the same in all directions. If we keep using the slices we defined, we can easily see that a manifold which has no

privileged directions about a point is called isotropic and it should be spherically symmetric about that point.
Assuming the cosmological principal holds, we first wish to describe the geometrical properties of space-times compatible with it. It turns out that all homogeneous and isotropic space-times can be described in terms of the Friedman-Robertson-Walker (FRW) line element.

$$ds^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right) \qquad (2)$$

Where k is the spatial curvature, scaled so as to take the values 0 or ± 1 .The case k=0 represents flat space and other two cases are space of constant positive or negative curvature, respectively.
The quantity a(t) is called the cosmic scale factor describes the overall expansion of the universe as a function of time .An important consequence of the expansion is that light from distance sources suffers a cosmological red shift as it travels along a null geodesic in the space time: ds = 0 in equation(1). If light emitted at time t_e is received by an observer at t_o then the red shift z of the source is given by

$$1 + z = \frac{a(t_o)}{a(t_e)} \qquad (3)$$

THE FRIEDMAN EQUATIONS

The dynamics of an FRW universe are determined by the Einstein gravitational field equations, which can be written, in tensor notation in the form

$$G_\mu^\nu = 8\pi G T_\mu^\nu \qquad (4)$$

Where T_μ^ν is the energy-momentum tensor describing the contents of the universe.
With $\rho = T_0^0$, $p = -\frac{1}{3} T_\alpha^\alpha$. The Einstein equation then simplify to

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho - \frac{3k}{a^2} + \Lambda\,, \qquad (5)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \qquad (6)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \qquad \text{(Taking c=1)} \qquad (7)$$

These equations determine the time evolution of the cosmic scale factor a(t) and therefore describe the global expansion or contraction of the universe. In the early phase of the big-bang, the universe is dominated by radiation for which $p = \frac{\rho}{3}$, while for late times it is matter- dominated so that $p \sim 0$ .The crossover between these two regimes occurs when the scale factor a is between 10^{-3} and $10^{-5}$of its present value, depending on the density of matter.
It is a property of the homogeneous and isotropic expansion of the universe around every point that these models can easily reproduce Hubble's law for the recession of galaxies

$$v = H_0\, r \qquad (8)$$

Where r is the distance of galaxy and v is recession velocity. The parameter H_0 is called the Hubble constant, in term of the scale factor

$$H_0 = \frac{\dot{a}}{a} \qquad (9)$$

The actual value of H_0 is not known with any great accuracy, but it probably 40km $s^{-1}Mpc^{-1} < H_0 < 90$ km $s^{-1}Mpc^{-1}$.

NEWTONIAN "DERIVATION" OF FRIEDMAN EQUATIONS

Consider a sphere, which expands in a homogeneous universe for non –relativistic particle the mass inside the sphere is constant.

Consider Model a piece of the universe as a ball of radius R and mass $M = \frac{4}{3}\pi G\rho$. Consider a small mass m attracted by this ball .Conservation of the kinetic plus potential energy of the small mass m implies.

$$\frac{1}{2}\ m\dot{a}^2 - \frac{G\,M\,m}{a} = -\ \frac{1}{2}\,k\ mc^2 \qquad (10)$$

Where the quantity on the right is some constant whose value is not determined by this Newtonian treatment, but which GR implies is given .

The energy equation (1) rearranges to

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \qquad (11)$$

Which reproduces the first Friedman equation.

FRIEDMANN EQUATION FROM FIRST LAW OF THERMODYNAMICS

For adiabatic expansion, the first law of thermodynamics is

$$dE + P\,dV = 0 \qquad (12)$$

With E= ρV and $V = \frac{4}{3}\pi a^3$, the first law (12) becomes

$$d(\rho a^3) + P\,da^3 = 0 \qquad (13)$$

Or, with the derivative taken with respect to cosmic time t

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \qquad (14)$$

Differentiating the first Friedman equation in the form

$$\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - kc^2 \qquad (15)$$

Gives $$2\ \dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho}\ a^2 + 2a\dot{a}\rho\right) \qquad (16)$$

And substituting $\dot{\rho}$ from the equation (7) reduces this to

$$2\ \dot{a}\ddot{a} = \frac{8\pi G}{3}\left(-3(\rho + p)\ \frac{\dot{a}}{a}\ a^2 + 2a\dot{a}\rho\right)$$

$$= \frac{8\pi G}{3} a\dot{a}\left(-(\rho + 3p)\right)$$

Which reproduce the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad (17)$$

SOLUTION OF FRIEDMANN EQUATION

In this paper we are going to solve Friedman equations for the matter dominated and radiation dominated universe and obtain the form of the scale factor a(t).we will also estimate the age of the flat Friedman universe .

We have Hubble constant

$$H = \frac{\dot{a}}{a} \qquad (18)$$

$$\dot{H} = -H^2 + \frac{\ddot{a}}{a}$$

$$= -H^2\left(1 - \frac{\ddot{a}}{H^2 a}\right)$$

$$\dot{H} = -H^2(1 + q) \qquad (19)$$

Where q(deceleration parameter) = $-\frac{\ddot{a}}{H^2 a}$ (20)

Matter-dominated universe:

It is modeled by dust approximation p = 0

Putting p =0 in equation (17) we have

$$\frac{\ddot{a}}{a}+\frac{4\pi G}{3}\rho=0$$

Using equation(20)

$$-H^2 q+\frac{4\pi G}{3}\rho=0 \qquad (21)$$

$$\rho=\frac{3H^2}{4\pi G}\,q \qquad (22)$$

Then the first Friedman equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2-\frac{8\pi G}{3}\rho=-\frac{k}{a^2}$$

$$H^2-2H^2 q=-\frac{k}{a^2} \qquad (23)$$

$$k=a^2H^2(2q-1) \qquad (24)$$

Since $a\neq 0$ and $H\neq 0$ for flat universe (k=o) equation (24) gives q= 1/2

Putting q=1/2 in equation (22) gives critical density

$$\rho_{cr}=\frac{3H^2}{8\pi G} \qquad (25)$$

$$=\frac{3\left(\frac{h}{98\times 10^{10}\ years}\right)^2\left(\frac{1\,year}{3600\times 24\times 365\,sec}\right)^2}{8\pi(6.67\times 10^{-8}\ c.m^3\ g^{-1}\ s^{-2}\ )}$$

$$=\ 1.87\times 10^{-29}\ h^2\ \frac{g}{c.m^3}$$

Using h$\approx$ $.72\pm .02$

$$\rho_{cr}\ \approx\ 10^{-29}\ \frac{g}{c.m^3}$$

From equation (22) and equation (25) gives relationship between density of the universe ρ and critical density ρ_{cr} as

$$q=\frac{\rho}{2\rho_{cr}} \qquad (26)$$

The equation (14) for the matter –dominated universe becomes

$$\dot{\rho}+3\rho\frac{\dot{a}}{a}=0$$

Multiplying by a^3

$$a^3\dot{\rho}+3\rho\dot{a}a^2=0$$

$$\frac{d}{dt}(a^3\rho)=0$$

$$a^3\rho=a_0^3\,\rho_0=constant \qquad (27)$$

For p =0 and $a^3\rho=constant$, the first Friedman equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho_o\left(\frac{a_0}{a}\right)^3$$

$$\frac{da}{dt}=\sqrt{\frac{8\pi\rho_0 G a_0{}^3}{3}}\ .\frac{1}{\sqrt{a}}$$

$$\sqrt{a}\ da=\sqrt{\frac{8\pi\rho_0 G a_0{}^3}{3}}\ dt$$

On integrating $\frac{2}{3}\ a^{3/2}=\sqrt{\frac{8\pi\rho_0 G a_0{}^3}{3}}\ t+C$ (28)

At the Big-Bang t=0, a=0, so C=0

Using $a_0 = 1 and\ C = 0$ equation (28) becomes

$a = (6\pi G\rho_0)^{1/3}\ t^{2/3}$, for the flat universe $\rho_0 = \rho_{cr}$

We have $a = (6\pi G\rho_{cr})^{1/3}\ t^{2/3}$

$= \left(6\pi G \frac{3H_0{}^2}{8\pi G}\right)^{1/3} t^{2/3}$

$= \left(\frac{9H_0{}^2}{4}\right)^{1/3} t^{2/3}$

$$=\left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \qquad (29)$$

$a \propto t^{2/3}$

Here we compute the age of the universe t_0, for Hubble rate H_0 scale factor $a = a_0 = 1$

We have $t_0 = \frac{2}{3H_0}$ (30)

Taking $H_0 = \frac{h}{0.98\times10^{10} years} and\ h \approx .72$

We get $t_0 = \frac{2\times0.98\times10^{10} years}{3\times0.72}$

$\approx 9.1 \times 10^9\ \ years$

$\approx 9.1\ \ (aeon)$ (31)

Radiation-dominated universe:

It is modeled by perfect fluid approximation $p = \frac{\rho}{3}$ the second Friedman equation (14) becomes

$\dot{\rho} + 3(\rho + \frac{\rho}{3})\frac{\dot{a}}{a} = 0$

$\dot{\rho} + 4\rho\frac{\dot{a}}{a} = 0$

Multiplying by a^4 we have $a^4\dot{\rho} + 4\rho\dot{a}a^3 = 0$

$\frac{d}{dt}(a^4\rho) = o$ $\Rightarrow$ $a^4\rho = a_0^4\rho_0 = constant$

For $p = \frac{\rho}{3}$ and $a^4\rho = constant$

The first Friedman equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_o\left(\frac{a_0}{a}\right)^4 \qquad (32)$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi\rho_o G a_0{}^4}{3}}\ .\frac{1}{a}$$

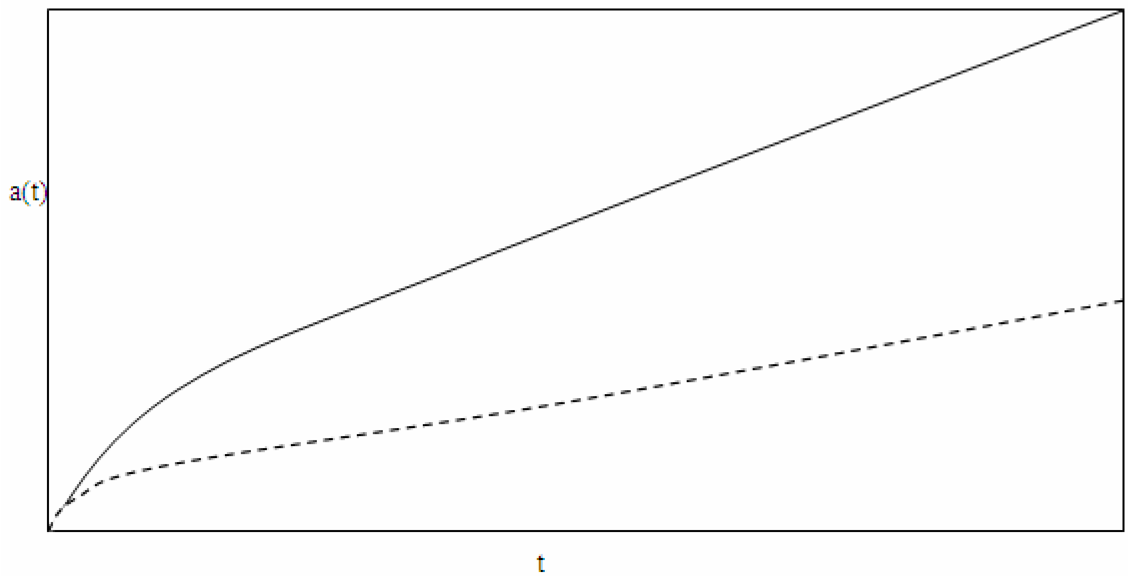
$$a\, da = \sqrt{\frac{8\pi\rho_0 G a_0{}^4}{3}}\ dt$$

On integrating we have $\frac{a^2}{2} = \sqrt{\frac{8\pi\rho_0 G a_0{}^4}{3}}\ t$ (33)

Or $a \propto t^{1/2}$ radiation

Which we can compare with

$a \propto t^{2/3}$



Matter –dominated ____________________
Radiation-dominated -------------------------

Figure 1: Evolution of the scale factor a (t) for the flat Friedman Universe.

CONCLUSION

In this paper we obtain derivation of Friedman equation in two different way, we observed that in the early universe the dominant energy density is that due to the radiation within the universe . The Friedman equation that was described in this paper is solved for such condition and the way in which the scale factor varies with time for such model is $a \propto t^{1/2}$ and for matter dominated universe is $a \propto t^{2/3}$.

It is not known at present whether the universe is open or closed. Now comparing average density of universe with a certain critical density. if average density is more than the critical density , the attractive force of different part of the universe towards each other will be enough to halt the recession eventually and to pull the galaxies together . If average density is less than the critical density the attractive force is insufficient and the expansion will continue forever.

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