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BIANCHI TYPE-I HOMOGENEOUS COSMOLOGICAL MODELS IN EINSTEIN THEORY AND MODIFIED THEORY OF GENERAL RELATIVITY

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ABSTRACT

The problem of spatially homogeneous and an isotropic Bianchi type-I with perfect fluid distribution is considered in Einstein theory and its modified theory proposed by Barber, [1]. Physically realistic and isotropic false vacuum models in both the theories are obtained by taking $\rho = -p$ and also new models are obtained by taking $\rho = 2p$ corresponding to “gamma law equation of state”. Some physical and geometrical properties of the solutions are also discussed.

Key Words: Modified Theory, De-Generate Model, Perfect Fluid, Isotropic, Singularity etc.

1. INTRODUCTION

It is well known in the literature that Einstein theory of gravitation is a coordinate invariant theory. It serves as a basis for mathematical models of the universes. Many alternative theories and modification to Einstein’s theory have been proposed by authors from time to time to unify gravitation and matter fields in various forms. In an attempt to produce continuous creation theories Barber [1] modified Brans and Dicke theory and general relativity. Both the theories create the universe out of self contained gravitational and matter fields. The second theory of Barber is a modification of general theory of relativity to a variable G-theory and predicts local effects which are within the observational limits. In the second theory of Barber the scalar field ‘ ϕ ’ does not gravitate directly, but simply divides the matter tensor acting as a reciprocal to gravitational constant

G i.e., $G = \frac{1}{\phi}$. Till now, no author has been studying the consistency of Barber second theory vis-à-

vis to Einstein’s theory of general relativity for an isotropic Bianchi type-I space-time in presence of macro matter field.

Thus in this paper we are interested to study the Bianchi Type-I spatially homogeneous cosmological models in Einstein theory and Barber’s second theory when the source of the gravitational field is a perfect fluid characterised by the equation of state $\rho = -p$ and $\rho = 2p$. In both the cases the metric reduces to isotropic models.

2. Field Equations

Here we consider the space-time described by an anisotropic and homogenous Bianchi type-I metric

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \quad (1)$$

where A, B and C are functions of cosmic time ‘t’ only.

Case I: The field equations of Einstein theory are

$$G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \quad (2)$$

Research Article

where the energy momentum tensor T_{ij} of gravitating macro matter field represented by perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij} \quad (3)$$

together with

$$g^{ij} u_i u_j = -1 \quad (4)$$

where u_i , p and ρ are four-velocity vector of the perfect fluid, proper pressure and energy density respectively. In co-moving co-ordinate system the field equations (2) and (3) for the metric (1) can be written as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -8\pi p, \quad (5)$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} = -8\pi p, \quad (6)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -8\pi p, \quad (7)$$

$$\text{and } \frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = 8\pi p, \quad (8)$$

where the index '4' denotes the ordinary differentiation with respect to time 't'.

The energy conservation equation of general relativity

$$T_{;j}^{ij} = 0 \quad (9)$$

takes the form

$$\rho_4 + (\rho + p) \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) = 0 \quad (10)$$

for the metric (1).

Case-II: The field equations of Barber's second self-creation theory are

$$G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij} = -8\pi \phi^{-1} T_{ij} \quad (11)$$

$$\text{and } \square \phi = \frac{8}{3} \pi \lambda T \quad (12)$$

where $\square \phi = \phi_{;k}^{ik}$ is the invariant D'Alembertian, T is the trace of energy momentum tensor T_{ij} , which is given in equation (3) and (4), λ is the coupling constant to be determined from experiment and ϕ is Barber's scalar. The measurement of the deflection of light restricts the value of the coupling to $0 < |\lambda| < 10^{-1}$. In the limit $\lambda \rightarrow 0$, the theory approaches the standard general theory of relativity in every respect.

In co-moving coordinate system the field equations (11) and (12) for the metric (1) can be obtained as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -8\pi \phi^{-1} p, \quad (13)$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} = -8\pi \phi^{-1} p, \quad (14)$$

Research Article

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -8\pi\phi^{-1}p, \quad (15)$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = 8\pi\phi^{-1}\rho, \quad (16)$$

$$\text{and } \phi_{44} + \phi_4 \left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right] = \frac{8\pi}{3} \lambda (\rho - 3p). \quad (17)$$

Here the index '4' denotes the ordinary differentiation with respect to time 't'.

We use a correspondence to general relativity and define equivalent densities and pressures (H.H. Soleng, [2]) as

$$\rho_{eq} = \frac{\rho}{\phi}, \quad p_{eq} = \frac{p}{\phi}. \quad (18)$$

Using equation (18) in equation (10), we get

$$\left(\frac{\rho}{\phi} \right)_4 + \left(\frac{\rho + p}{\phi} \right) \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) = 0. \quad (19)$$

3. Solutions and Models

In this section, we intend to find the cosmological models in both the theories. The field equation system (5) – (8) and (13) – (17) in section-2 are an underdetermined system. To make the system consistent, we use “gamma law equation of state”

$$p = \gamma\rho, \quad -1 \leq \gamma \leq 1 \quad (20)$$

as an additional condition.

3.1. False Vacuum Models

When $\gamma = -1$, equation (20) reduces to the form

$$\rho + p = 0 \quad (21)$$

which is known as ‘false vacuum’ or ‘de-sitter model’, Blome and Priester [3].

Case-I: In this case adding equations (5), (6) and (7) with three times of equation (8), we get

$$\frac{[ABC]_{44}}{ABC} = 24\pi\rho. \quad (22)$$

Using equation (21) in equation (10) and then integrating, we find

$$\rho = k \quad (23)$$

where k is a constant of integration.

Again using equation (23) in equation (22), we obtain

$$[ABC]_{44} = k_1^2 ABC \quad (24)$$

where $k_1^2 = 24\pi k$.

On integration, equation (24) yields

$$([ABC]_4)^2 = k_1^2 (ABC)^2 + k_2. \quad (25)$$

Here, to avoid the complication and to obtain the solution, we consider the integral constant $k_2 = 0$.

Thus the solution of the equation (25) is

$$ABC = e^{k_1 t + k_3}. \quad (26)$$

Substituting $k_1 t + k_3 = T$ in equation (26), we get

$$ABC = e^T. \quad (27)$$

Research Article

From equation (27), the explicit form of A, B and C are

$$A = e^{Tn_1}, B = e^{Tn_2} \text{ and } C = e^{Tn_3} \quad (28)$$

where n_i , $i = 1, 2, 3$ are real constants satisfying the condition

$$\sum_{i=1}^3 n_i = 1. \quad (29)$$

Here the over determinacy for obtaining the unknowns A, B and C from four equations (5) – (8) can be settled by actual substitution of solution (28) in the field equations. The additional condition is obtained as

$$\sum_{\substack{i,j=1 \\ i < j}}^3 n_i n_j = \frac{1}{3}. \quad (30)$$

Further equation (29) and (30) yields

$$\sum_{i=1}^3 n_i^2 = \frac{1}{3} \text{ and } n_1 = n_2 = n_3 = \frac{1}{3}. \quad (31)$$

Thus subject to restriction given in equation (30), the explicit expression for A, B and C are

$$A = B = C = e^{\frac{T}{3}}. \quad (32)$$

Thus anisotropic homogenous cosmological model is reduced to false vacuum isotropic homogeneous cosmological model in Einstein theory of gravitation and that is

$$ds^2 = -dT^2 + e^{\frac{2T}{3}} [dx^2 + dy^2 + dz^2]. \quad (33)$$

Case-II: In this case adding equations (13), (14) and (15) with three times of equation (16), we get

$$\frac{[ABC]_{44}}{ABC} = 24\pi \frac{\rho}{\phi}. \quad (34)$$

Using equation (21) in equation (19) and then integrating, we find

$$\frac{\rho}{\phi} = k_4 \quad (35)$$

where k_4 is a constant of integration.

Using equation (35) in equation (34), we obtain

$$\frac{[ABC]_{44}}{ABC} = 24\pi k_4 \quad (36)$$

$$\text{i.e. } [ABC]_{44} = k_5^2 ABC$$

$$\text{where } k_5^2 = 24\pi k_4.$$

On integration, equation (36) yields

$$([ABC]_4)^2 = k_5^2 (ABC)^2 + k_6. \quad (37)$$

In order to obtain solution, we consider the integral constant $k_6 = 0$.

Thus the solution of the equation (37) is

$$ABC = e^{k_5 t + k_7} \quad (38)$$

where k_7 is the constant of integration.

Substituting $k_5 t + k_7 = T$ in equation (38), we get

Research Article

$$ABC = e^T. \quad (39)$$

From equation (39), the explicit form of A, B and C are

$$A = e^{Tp_1}, B = e^{Tp_2} \text{ and } C = e^{Tp_3}, \quad (40)$$

where p_i , $i = 1, 2, 3$ are real constants satisfying the condition

$$\sum_{i=1}^3 p_i = 1. \quad (41)$$

Here the over determinacy for obtaining the unknowns A, B and C from four equations (13) – (16) can be settled by actual substitution of solution (40) in the field equations. Thus the additional condition is obtained as

$$\sum_{\substack{i,j=1 \\ i < j}}^3 p_i p_j = \frac{1}{3}. \quad (42)$$

Further equation (41) and (42) yields to

$$\sum_{i=1}^3 p_i^2 = \frac{1}{3} \text{ and } p_1 = p_2 = p_3 = \frac{1}{3}. \quad (43)$$

Thus subject to restriction given in equation (42), the explicit expression for A, B and C are

$$A = B = C = e^{\frac{T}{3}}. \quad (44)$$

With the use of equation (21), equation (17) reduces to

$$\phi_{44} + \phi_4 \left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right] = \frac{32}{3} \pi \lambda \rho. \quad (45)$$

Using equation (35) and (38) in equation (45), we obtain

$$\phi_{44} + k_5 \phi_4 = \frac{32}{3} \pi \lambda k_4 \phi. \quad (46)$$

Now equation (46) in T co-ordinates, reduces to

$$\phi_{TT} + \phi_T - \frac{4}{9} \lambda \phi = 0. \quad (47)$$

On integration, equation (47) yields

$$\phi = c_1 e^{q_1 T} + c_2 e^{q_2 T} \quad (48)$$

$$\text{where } q_1 = \frac{-3 + \sqrt{9 + 16\lambda}}{6} \text{ and } q_2 = \frac{-3 - \sqrt{9 + 16\lambda}}{6}.$$

Using equation (48) in equation (35), we obtain

$$\rho(-p) = k_4 \phi = k_4 (C_1 e^{q_1 T} + C_2 e^{q_2 T}). \quad (49)$$

Thus anisotropic homogeneous cosmological model is reduced to isotropic homogeneous cosmological false vacuum model in Barber's second theory of gravitation and that is

$$ds^2 = -dT^2 + e^{\frac{2T}{3}} [dx^2 + dy^2 + dz^2]. \quad (50)$$

3.2. When $\gamma = \frac{1}{2}$, equation (20) reduces to the form

$$\rho = 2p \quad (51)$$

which is purely a new model.

Research Article

Case I : In this case adding equation (5), (6) and (7) with three times of equation (8), we get

$$\frac{[ABC]_{44}}{ABC} = 6\pi\rho. \quad (52)$$

Using equation (51) in equation (10), we obtain

$$\frac{2}{3} \frac{\rho_4}{\rho} + \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} = 0. \quad (53)$$

Integrating equation (53), we obtain

$$\rho = \left(\frac{K_8}{ABC} \right)^{\frac{3}{2}}. \quad (54)$$

Using equation (54) in equation (52) and integrating we obtain

$$([ABC]_4)^2 = 24 K_8^{\frac{3}{2}} \pi (ABC)^{\frac{1}{2}} + K_9 \quad (55)$$

where K_9 is the constant of integration.

To avoid the complication and to obtain the solution, we consider $K_9 = 0$ in (55) and then we get

$$[ABC]_4 = K_{10} (ABC)^{\frac{1}{4}}, \quad K_{10} = \sqrt{24 K_8^{\frac{3}{2}} \pi}. \quad (56)$$

Again integrating equation (56), we obtain

$$ABC = \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{3}}. \quad (57)$$

From equation (57), the explicit form of A, B, and C can be taken as

$$\left. \begin{aligned} A &= \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{3} m_1}, \\ B &= \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{3} m_2}, \\ C &= \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{3} m_3} \end{aligned} \right\} \quad (58)$$

where $m_i, i = 1, 2, 3$ are real constants satisfying the condition

$$\sum_{i=1}^3 m_i = 1. \quad (59)$$

Substituting the solutions of equation (58) in equations (5) – (8) and using equation (59), the additional conditions are obtained as

$$\sum_{i=1}^3 m_i^2 = \frac{1}{3} \quad \text{and} \quad \sum_{\substack{i,i=1 \\ i \neq j}}^3 m_i m_j = \frac{1}{3}$$

which yields

$$m_1 = m_2 = m_3 = \frac{1}{3}.$$

Thus the explicit expression for A, B and C are

Research Article

$$A = B = C = \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{9}}. \quad (60)$$

Hence the spatially isotropic and homogenous cosmological model which is a new model taking $\rho=2p$ in Einstein theory of gravitation is designed as

$$ds^2 = -dt^2 + \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{9}} (dx^2 + dy^2 + dz^2). \quad (61)$$

Case II : In this case adding equation (13), (14) and (15) with three times equation (16), we get

$$\frac{[ABC]_{44}}{ABC} = 6\pi \frac{\rho}{\phi}. \quad (62)$$

Using equation (51) in equation (19) and integrating, we find

$$\frac{\rho}{\phi} = \left(\frac{k_{12}}{ABC} \right)^{\frac{3}{2}} \quad (63)$$

where k_{12} is the constant of integration.

Now using equation (63) in equation (62) and integrating we obtain

$$([ABC]_4)^2 = 24 k_{12}^{\frac{3}{2}} \pi (ABC)^{\frac{1}{2}} + k_{13} \quad (64)$$

where k_{13} is the constant of integration. To avoid the complication and to obtain the solution, we consider $k_{13} = 0$ in equation (64) and then we obtain

$$[ABC]_4 = k_{14} (ABC)^{\frac{1}{4}} \quad (65)$$

where $k_{14} = \sqrt{24 k_{12}^{\frac{3}{2}} \pi}$.

Again integrating equation (65), we get

$$ABC = \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{3}} \quad (66)$$

where k_{15} is the constant of integration.

From equation (66), the explicit form of A, B and C can be taken as

$$\left. \begin{aligned} A &= \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{3} q_1}, \\ B &= \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{3} q_2}, \\ C &= \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{3} q_3} \end{aligned} \right\} \quad (67)$$

where q_i ; $i = 1, 2, 3$ are real constants satisfying the condition

$$\sum_{i=1}^3 q_i = 1. \quad (68)$$

Substituting the solutions of equation (67) in equations (13) – (16) and using equation (68), the additional conditions are obtained as

Research Article

$$\sum_{i=1}^3 q_i^2 = \frac{1}{3} \quad \text{and} \quad \sum_{\substack{i,j=1 \\ i \neq j}}^3 q_i q_j = \frac{1}{3}$$

which yield

$$q_1 = q_2 = q_3 = \frac{1}{3}.$$

Thus the explicit expression for A, B and C are

$$A = B = C = \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{9}}. \quad (69)$$

With the case of equation (51), equation (17) reduces to

$$\phi_{44} + \phi_4 \left[\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right] = -\frac{4\pi\lambda\rho}{3}. \quad (70)$$

Using equation (63) and (66) in equation (70), we obtain

$$\left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^2 \phi_{44} + k_{14} \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right) \phi_4 + \frac{4\pi\lambda k_{12}^{\frac{3}{2}}}{3} \phi = 0. \quad (71)$$

Solving equation (71), the solution is obtained as

$$\begin{aligned} \phi = \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{-\frac{1}{6}} & \left[C_1 \cos \frac{\sqrt{12 k_{16} - 1}}{6} \ln \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right) \right. \\ & \left. + C_2 \sin \frac{\sqrt{12 k_{16} - 1}}{6} \ln \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right) \right] \end{aligned} \quad (72)$$

where k_{16} is the constant of integration.

Now using equation (72) in equation (63), we obtain

$$\begin{aligned} \rho (= 2p) = k_{12}^{\frac{3}{2}} \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{-\frac{13}{6}} & \left[C_1 \cos \frac{\sqrt{12 k_{16} - 1}}{6} \ln \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right) \right. \\ & \left. + C_2 \sin \frac{\sqrt{12 k_{16} - 1}}{6} \ln \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right) \right]. \end{aligned} \quad (73)$$

Hence the spatially isotropic and homogeneous cosmological model (taking $\rho = 2p$) which is a new model in barber's second theory of gravitation is designed as

$$ds^2 = -dt^2 + \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{9}} [dx^2 + dy^2 + dz^2]. \quad (74)$$

4. Physical and Geometrical Properties

4.1 In section 3, we have shown that equation (33) and (50) represent the false vacuum dominated universes in both the theories of gravitation. The physical parameters involved in both the models (33) and (50) behave as follows:

(i) As $T \rightarrow 0$, $\phi \rightarrow$ a constant and $\rho (= -p) \rightarrow$ constant. In this case both the space-times reduce to flat space-times.

(ii) The magnitude of scalar expansion θ for both the models are constants. Hence both the universes have uniform expansion.

Research Article

(iii) The spatial volume in both the models is of the form $V = e^T$. Here $V \rightarrow 1$ as $T \rightarrow 0$ and $V \rightarrow \infty$ as $T \rightarrow \infty$. Thus it is inferred that the universe starts with unit volume and blows up at infinite future.

(iv) The magnitude of shear σ for both the models is $\sigma^2 = 0$. This indicates that the universes remain isotropic and non-shearing throughout the evolution. Also $\lim_{t \rightarrow \infty} \frac{\sigma}{\theta} = 0$ in both the cases

confirm that universes are isotropic in nature.

(v) Also the vorticity tensor (ω) for both the models is zero, which show that universes are non-rotating.

(vi) The Kretschmann curvature invariants are found to be

$$L = \frac{5k_1^4}{27e^{4T}} \text{ for the model (33)}$$

$$\text{and } L = \frac{5k_5^4}{27e^{4T}} \text{ for the model (50).}$$

It clearly shows in both the cases that the universes possess no geometrical singularities at $T = 0$.

4.2 The physical and geometrical properties of the universes (61) and (74)

(i) The volume of the universe (61) is

$$V = \left(\frac{3}{4} k_{10} t + \frac{3}{4} k_{11} \right)^{\frac{4}{3}}.$$

$$\text{Hence } V = \left(\frac{3}{4} k_{11} \right)^{\frac{4}{3}} \text{ as } t \rightarrow 0 \text{ and } V \rightarrow \infty \text{ as } t \rightarrow \infty.$$

The volume of the universe (74) is

$$V = \left(\frac{3}{4} k_{14} t + \frac{3}{4} k_{15} \right)^{\frac{4}{3}}.$$

$$\text{Here } V = \left(\frac{3}{4} k_{15} \right)^{\frac{4}{3}} \text{ as } t \rightarrow 0 \text{ and } V \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Thus it is inferred that both the universes start with a constant volume and blows up at infinite future.

(ii) The magnitude of Scalar expansion θ for both the universes are derived as

$$\theta = \frac{k_{10}}{\frac{3}{4} k_{10} t + \frac{3}{4} k_{11}} \text{ for model (61)}$$

$$\text{and } \theta = \frac{k_{14}}{\frac{3}{4} k_{14} t + \frac{3}{4} k_{15}} \text{ for model (74)}$$

In both the cases the universes are expanding but the rate of expansion is slow with increase of time.

(iii) The magnitude of the shear σ is $\sigma^2 = 0$ for both the universes (61) and (74). It indicates that the universes are non-shearing and isotropic in nature.

(iv) The Kretschmann curvature invariants are found to be

$$L = \frac{2281}{3^7} \left(\frac{k_{10}}{k_{10} t + k_{11}} \right)^4 \text{ for the model (61)}$$

Research Article

$$\text{and } L = \frac{2281}{3^7} \left(\frac{k_{14}}{k_{14}t + k_{15}} \right)^4 \text{ for model (74)}$$

It clearly shows that both the universes possess no singularity at $t = 0$.

5. Conclusion

In this paper, we have obtained the important cosmological models i.e. false vacuum model by taking $\rho = -p$ and new type of models by taking $\rho = 2p$ in both Einstein and Barber's second theory. It is observed that the models in both the theories are expanding uniformly, non-rotating, non-shearing, isotropic and have no geometrical singularities.

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