International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online International Journal Available at http://www.cibtech.org/jpms.htm
2012 Vol. 2 (1) January-March, pp.214-220/ Shalini Gupta.
Research Article

SOME REMARKS ON THE METHODS OF CONSTRUCTION OF DYNAMICAL INVARIANTS

***Shalini Gupta**
Department of Physics, H.M.R Institute of Technology and Management Hamidpur, Delhi-110036 India,
**Author for Correspondence*

ABSTRACT
For forced time dependent harmonic oscillator a comparative study of different methods for the construction of invariants is carried out. Inspite of their different procedural details all the approaches lead to the same invariant for the given classical system. Limitations of different methods are briefly highlighted.

Key Words: *Invariants, Forced Harmonic Oscillator, etc.*
PACS NO.: 05.45-a

INTRODUCTION
An invariant of a dynamical system is a single valued phase space function (Kaushal1998)**,** which is in involution with the Hamiltonian and with other invariants, if they exist, for the system. A system is said to be integrable if it is possible to obtain all functionally independent invariants or constants of motion which are in involution among themselves and with the Hamiltonian of the system. For the systems involving explicit time dependence, however, one also looks for an additional invariant as its Hamiltonian is no longer a constant of motion.

With regard to the construction of dynamical invariants several methods have been developed in the past (Lutzky1978,Hietarinta1987,Hall1983) and they are employed to study (Kaushal1998) the systems in one and higher dimensions, however, at times without noticing the underlying intricacies and the limitations of the methods. As a matter of fact, it is found that every method does not successfully work for every system. Further, these methods, while originating from different mathematical roots, interestingly, are found to provide the same result at least for certain systems. Sometimes, the reproduction of results for these latter systems becomes the testing ground for the newly developed method. One such case is that of time dependent (TD) harmonic oscillator or its variants in one dimension, first studied by Lewis (Lewis1968).

With a view to highlight the underlying limitations and intricacies of some of the popular methods used in the past, here in the present work we investigate the example of a 1-D,TD, forced harmonic oscillator corresponding to a potential term, $V(x,t)=1/2\ \omega^2(t)x^2 - f(t)x$, in the Hamiltonian. In particular, the methods we discuss, are the following:

a. Dynamical algebraic approach (Korsch1979,Kaushal and Korsch198)]
b. Lutzky's approach using Noether's theorem (Lutzky1978)
c. Transformation group approach (Ray1979,1980,Burgan et al1979)
d. Ermakov's approach (Ermakov1880)
e. Rationalization method (Kaushal1998;Whittaker1927)

Construction of the Invariant
a. Dynamical Algebraic Approach: Korsch (Korsch1979)and Kaushal and Korsch (Kaushal and Korsch1981) employed this approach to construct the invariants for a variety of TD systems. This algebraic technique (Takayama1982) provides a direct and unsophisticated derivation of dynamical invariants but only for a limited class of TD potentials. However, it allows a straight-forward transition from classical to quantum systems because in algebraic treatment the formulation of classical and quantum dynamics is almost identical except for the fact that Poisson bracket is now replaced by a commutator and dynamical variables become operators.

For the present case of TD, forced harmonic oscillator one writes the Hamiltonian as

$$H = \frac{1}{2}(p^2 + \omega^2(t)x^2 - 2f(t)x) \tag{1}$$

Next, one applies the dynamical algebraic approach [Kaushal and Korsh1981] by identifying H in (1) as

$$H = h_1\Gamma_1 + h_2\Gamma_2 + h_3\Gamma_3 \tag{1$'$}$$

where $h_1 = 1, h_2 = \omega^2(t), h_3 = -f, h_4 = h_5 = h_6 = 0$

and the phase space functions, $\Gamma_i{}'$ s as

$$\Gamma_1 = \frac{1}{2}p^2; \Gamma_2 = \frac{1}{2}x^2; \Gamma_3 = x; \Gamma_4 = -xp; \Gamma_5 = -p; \Gamma_6 = 1.$$

Since the invariant I is also a phase space function and so is the member of the Lie algebra, it should be expressible as

$$I = \Sigma_k \lambda_k(t)\,\Gamma_k \tag{2}$$

and satisfies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0 \tag{3}$$

Further, note that the phase space functions Γ_3, Γ_4, Γ_6 in this case are needed to close the algebra and are chosen mainly out of intuition. Further, in view of the closure property of the Lie algebra, the non vanishing Poisson brackets turnout to be

$$[\Gamma_1, \Gamma_2] = \Gamma_4 \; ; \; [\Gamma_1, \Gamma_3] = \Gamma_5 \; ; \; [\Gamma_1, \Gamma_4] = 2\Gamma_1 \; ; \; [\Gamma_2, \Gamma_4] = -2\Gamma_2 \tag{4}$$
$$[\Gamma_2, \Gamma_5] = -\Gamma_3 \; ; \; [\Gamma_3, \Gamma_4] = -\Gamma_3 \; ; \; [\Gamma_3, \Gamma_5] = -\Gamma_6 \, .$$

As per prescription of this method, we use

$$[\Gamma_n, \Gamma_m]_{PB} = \sum_r C^r_{nm}\Gamma_r \qquad \text{and} \qquad \dot{\lambda}_r - \sum_n \left[\sum_m C^r_{nm}\, h_m(t)\right]\lambda_n = 0$$

to obtain the following set of first-order coupled differential equation

$$\dot{\lambda}_1 = 2\lambda_4; \dot{\lambda}_2 = -2\omega^2(t)\lambda_4; \dot{\lambda}_3 = f\lambda_4 - \omega^2(t)\lambda_5$$
$$\dot{\lambda}_4 = \lambda_2 - \omega^2(t)\lambda_1; \dot{\lambda}_5 = \lambda_3 + f\lambda_1; \dot{\lambda}_6 = -f\lambda_5 \, .$$

Assuming $\lambda_5 = \psi(t)$ and $\lambda_1 = \rho^2(t)$, one immediately finds other λ's as

$$\lambda_2 = \dot{\rho}^2 + \frac{K}{\rho^2}; \lambda_3 = \dot{\psi} - \rho^2 f; \lambda_4 = \rho\dot{\rho}; \lambda_6 = \int^t \psi(\tau) f(\tau) d\tau \, .$$

Finally, the invariant for the forced harmonic oscillator turns out to be

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \frac{K}{2}\left(\frac{x}{\rho}\right)^2 + \dot{\psi}x - \psi\dot{x} + \int^t \psi(\tau) f(\tau) d\tau - \rho^2 x f \quad , \tag{5}$$

where $\rho(t)$ and $\psi(t)$ are arbitrary functions satisfying the auxillary equations

$$\ddot{\rho} + \omega^2(t)\rho = \frac{K}{\rho^3} \quad ; \tag{6}$$

and $$\ddot{\psi} + \omega^2(t)\psi = \rho^2\dot{f} + 3\rho\dot{\rho}f \, . \tag{7}$$

Eqs. (6) and (7) constitute an Ermakov system of equations [Ermakov1880], where K is some arbitrary constant. Note that in this particular example, the phase space function Γ_6 turns out to be a constant quantity.

b. Lutzky's Approach*:* Lutzky [Lutzky1978] suggested a method for constructing an invariant using the results of Noether's theorem for the Lagrangian $L(x, \dot{x}, t)$.In this approach, the symmetry transformation is described by the group operator

$$X = \xi(x,t)\frac{\partial}{\partial t} + \eta(x,t)\frac{\partial}{\partial x}$$

If the transformation described by the group operator X leaves the action $A = \int L(x, \dot{x}, t)\,dt$ invariant, then a certain combination of derivative terms gives rise to terms equal to the total time derivative of a function F, viz.,

$$\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L = \dot{F}(x,t) \qquad (8)$$

Finally, the constant of motion for the above system turns out to be

$$I = (\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + F \quad , \qquad (9)$$

where $\dot{\xi}$, $\dot{\eta}$ and $\dot{F}$ are defined as

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\dot{x}\ ;\ \dot{\eta} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\dot{x},\ \dot{F} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\dot{x} \ .$$

Now, corresponding to a forced harmonic oscillator consider the Lagrangian

$$L = \frac{1}{2}\left[\dot{x}^2 - \omega^2(t)x^2 + 2f(t)x\right] , \qquad (10)$$

with concomitant equation of motion

$$\ddot{x} + \omega^2(t)x = f(t) \ . \qquad (11)$$

For this form of L, eq. (8) yields

$$\left(-\omega\dot{\omega}\xi - \dot{\xi}\omega^2 - \frac{1}{4}\dddot{\xi}\right)x^2 - \left[(\ddot{\xi} + \omega^2(t)\psi) - (\xi\dot{f} + \dot{f}f)\right]x + \frac{1}{2}fx\dot{\xi} + f\phi - \dot{\chi} = 0 \,,$$

where $\xi(t)$, $\psi(t)$ and $\chi(t)$ are arbitrary functions of t. For the equations of motion to have Noether's symmetry, the expression (11) can be rationalized in the powers of x, implying

$$\dddot{\xi} + 4\omega^2\dot{\xi} + 4\omega\dot{\omega}\xi = 0 \ ;$$

$$\ddot{\psi} + \omega^2(t)\psi = \xi\dot{f} + \frac{3}{2}\dot{\xi}f$$

$$\dot{\chi} = \psi(t)f(t) \,.$$

Through the transformation $\xi = \rho^2(t)$, the above set of equations immediately yields

$$\ddot{\rho} + \omega^2(t)\rho = \frac{K}{\rho^3} \ ; \qquad (12a)$$

$$\ddot{\psi} + \omega^2(t)\psi = \rho^2\dot{f} + 3\rho\dot{\rho}f \qquad (12b)$$

$$\chi = \int^t \psi(\tau)f(\tau)d\tau \ . \qquad (12c)$$

Finally the Noether invariant (9) takes the form

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \dot{\psi}x - \psi\dot{x} + \int^{t}\psi(\tau)f(\tau)d\tau - \rho^2 xf + \frac{K}{2}\left(\frac{x}{\rho}\right)^2 \qquad (13)$$

which is exactly the same as obtained by using dynamical algebraic approach.

c. Transformation-Group Method: Ray (Ray1979,1980)constructed the invariant for a forced. TD harmonic oscillator by using transformation-group technique. This method suggested by Burgan et al (Burgan et al 1979) transforms the equation of motion (11) via the transformation

$$\begin{aligned} x' &= \frac{x}{\rho(t)} + A(t) \\ t' &= D(t) \end{aligned} \qquad (14)$$

where ρ(t), A(t) and D(t) are arbitrary functions, leading to the form

$$\rho\dot{D}^2\frac{d^2x'}{dt'^2} + (2\dot{\rho}\dot{D} + \rho\ddot{D})\frac{dx'}{dt'} + (\ddot{\rho} + \omega^2(t)\rho)x'$$
$$. + (-\ddot{\rho}A - 2\dot{\rho}\dot{A} - \omega^2(t)\rho A - \rho\ddot{A} - f) = 0 \qquad (15)$$

In order that the form of this equation remains invariant under the group transformation (14), one should have

$$2\dot{\rho}\dot{D} + \rho\ddot{D} = 0 \quad or \quad \dot{D} = \left(\frac{1}{\rho^2}\right) \qquad (16)$$

and

$$\frac{1}{\rho^3}\frac{d^2x'}{dt'^2} + (\ddot{\rho} + \omega^2(t)\rho)x' - (\ddot{\rho}A + 2\dot{\rho}\dot{A} + \omega^2(t)\rho A + \rho\ddot{A} + f) = 0\,. \qquad (17)$$

To simplify eq.(15), we set the arbitrary functions ρ(t) and A(t) such that

$$\ddot{\rho}(t) + \omega^2(t)\rho = K/\rho^3\,, \qquad (18)$$

$$\rho^4\ddot{A} + KA + 2\dot{\rho}\dot{A}\rho^3 = -\rho^3 f\,. \qquad (19)$$

The last expression reduces to the relation

$$\ddot{\psi} + \omega^2\psi = \rho^2\dot{f} + 3\rho\dot{\rho}f\,, \qquad (20)$$

if $\dot{A} = -\psi/\rho^3\,.$ (21)

Finally, the equation of motion in the primed coordinates turns out to be

$$\frac{d^2x'}{dt'^2} + kx' = 0\,,$$

and the invariant I, after inverse transformation, reduces to the form

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x + \rho^2\dot{A})^2 + \frac{K}{2}\left(\frac{x}{\rho} + A\right)^2\,, \qquad (22)$$

where, ρ is any solution to (18) and A is any solution to (19).
Using (20) and (21) the above invariant reduces to

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \dot{\psi}x - \psi\dot{x} + \int^{t}\psi(\tau)f(\tau)d\tau) - \rho^2 xf + \frac{K}{2}\left(\frac{x}{\rho}\right)^2\,. \qquad (23)$$

which again has the same mathematical structure as in eq. (13) or (5)

d. Ermakov Method: Ray and Reid [Ray and Reid1971,1979,1982] and several others(see for example the references cited in [Kaushal1998]) have applied this method to a variety of TD oscillator problems and their possible generalizations. In their course of study they have evolved a method of constructing the invariant for 1-D, TD systems known as Ermakov method and the invariant so constructed is termed as **Ermakov invariant** in the literature. Consider the system of forced TD harmonic oscillator described by eq. (11) with auxiliary equations

$$\ddot{\rho} + \omega^2(t)\rho = \frac{K}{\rho^3} , \qquad (24)$$

and. $$\ddot{\psi} + \omega^2(t)\psi = \rho^2 \dot{f} + 3\rho\dot{\rho}f \qquad (25)$$

Now eliminate $\omega^2(t)$ from eqs. (11) and (24) and multiply the resultant expression by the factor, $(\rho\dot{x} - \dot{\rho}x)$ a simplification of various terms in the resultant expression leads to

$$\frac{1}{2}\frac{d}{dt}(\rho\dot{x} - \dot{\rho}x)^2 + \frac{K}{2}\frac{d}{dt}\left(\frac{x}{\rho}\right)^2 = (\rho\dot{x} - \dot{\rho}x)\rho f \qquad (26)$$

Next, eliminate $\omega^2(t)$ from (11) and (25) and rearranging the terms in the resulting equation, one obtains

$$\frac{d}{dt}(\dot{x}\psi - \dot{\psi}x) + \frac{d}{dt}(\rho^2 fx) - f\psi = (\rho\dot{x} - \dot{\rho}x)\rho f , \qquad (27)$$

A comparison of eqs. (26) and (27) immediately yields the invariant

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \frac{K}{2}\left(\frac{x}{\rho}\right)^2 - \rho^2 fx + \dot{\psi}x - \psi\dot{x} + \int^t \psi(\tau) f(\tau) d\tau , \qquad (28)$$

which is exactly of the same form as obtained in the three previous methods. In fact, due to the presence of two TD functions in the equation of motion (11), two auxiliary equations are required here whereas in the absence of the forcing term in (10) there remains only one auxiliary equation.

e. Rationalization Method: In the rationalization method, one normally considers an ansatz for the n^{th} order invariant for a 1-D, TD system [Kaushal1998] as

$$I = a_0 + a_1\dot{x} + \frac{1}{2!}a_2\dot{x}^2 + ...+ \frac{1}{n!}a_n\dot{x}^2 , \qquad (29)$$

where the coefficient functions $a_i=a_i(x,t)$.For the Hamiltonian $H = \frac{1}{2}p^2 + V(x,t)$, while the details of the method for an arbitrary potential in general can be found in [Kaushal1998], we concentrate here on the construction of invariant for the system (1) and use the terms upto $\dot{x}^2$ in (29). To this effect, we use (29) upto $\dot{x}^2$ term in (3) and rationalize the resultant expression with respect to the powers of x. For the construction of second-order invariant in momentum this yields (Kaushal1998) the following set of coupled partial differential equations for the coefficients a_i's in (29)

$$\frac{\partial a_2}{\partial x} = 0 , \qquad (30a)$$

$$\frac{2\partial a_1}{\partial x} + \frac{\partial a_2}{\partial t} = 0 , \qquad (30b)$$

$$\frac{\partial a_0}{\partial x} + \frac{\partial a_1}{\partial t} - a_2\left(\frac{\partial V}{\partial x}\right) = 0 , \qquad (30c)$$

$$\frac{\partial a_0}{\partial t} - a_1\left(\frac{\partial V}{\partial x}\right) = 0 \quad . \tag{30d}$$

The integration of eq.(30a) and (30b) immediately yields

$$a_2 = \beta(t) \ , \ a_1 = -\frac{1}{2}\dot{\beta}(t)x - .\psi(t) \ . \tag{31}$$

where β and ψ are arbitrary functions of t. For V in (30),we use the form given in (1). Then, eliminating a_0(x,t) from eq.(30c) and (30d) and rationalizing the resultant expression in powers of x gives rise to two auxiliary equations (after identifying $\beta = \rho^2(\mathrm{t})$), namely

$$\ddot{\rho} + \omega^2(t)\rho = \frac{K}{\rho^3} \quad ,$$

$$\ddot{\psi} + \omega^2(t)\psi = \rho^2\dot{f} + 3\rho\dot{\rho}f \ .$$

The coefficient function a_0(x,t) can be determined from (30c) and (30d) as

$$a_0(x,t) = \frac{1}{2}(\rho\ddot{\rho} + \dot{\rho}^2)x^2 + \dot{\psi}x - \rho^2 fx + \frac{1}{2}\rho^2\omega^2 x^2 + \int^t \psi(\tau)f(\tau)d\tau \tag{32}$$

Finally, using (31) and (32) in expression (29) one obtains the invariant in the form

$$I = \frac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \dot{\psi}x - \psi\dot{x} + \int^t \psi(\tau)f(\tau)d\tau) - \rho^2 xf + \frac{K}{2}\left(\frac{x}{\rho}\right)^2 ,$$

which is exactly the same as demonstrated by other four approaches.

Viability of Different Methods

Different methods are used for the construction of invariants for 1-D, TD forced harmonic oscillator and all the methods lead to the same form of dynamical invariant.To this particular form of potential the viability of a formal method like rationalization method is demonstrated here for the first time. Although none of the methods has a universal character, in the recent years the Lie algebraic approach for construction of dynamical invariants has provided many interesting results (Kaushal and Mishra1993),(Kaushal and Gupta2001),(Kaushal et al 2001). It has additional advantage of having a straight forward application to quantum mechanical systems and classical systems in higher dimensions, yet its limitation in providing second invariant for 2- and 3-D, TD systems could not be ruled out. Ermakov method, for constructing invariant, is systematic and uses simple mathematics as used to derive constants of motion in mechanics. The weakness of Ermakov procedure is that we must know both the equation of motion for the system and the auxiliary equation in order to derive the invariant. As the auxiliary equations for the forced harmonic oscillator, Ref equation (6) and (7) are set in advance for an arbitrary form of V(x, t) ,however such a guess becomes a difficult task and one has to try the formal methods. However for a general equation of motion, auxiliary equations are not known and a different method is required.

Moreover in the Lie algebraic approach certain features of the equation of motion (like linearity or non-linearity) are retained through the closure property of the algebra. This is not the case in rationalization method which is most commonly used for TD systems. Here however the degree of complexity further increases as we move to higher dimensions.

CONCLUSION

Invariants if they exist can be very useful in studying the theoretical structure of dynamical systems. In this work we have demonstrated the viability of various methods using the example of forced TD harmonic oscillator. Inspite of so many methods for 1-D systems, not many new systems are found for

Research Article

which the invariants can be constructed. Moreover not all the methods used are yet applied to higher dimensional systems, where the underlying intricacies of a method are reflected much more.

ACKNOWLEDGEMENTS
I am grateful to the management of HMR Institute of Technology and Management for their encouragement and support. Thanks are also due to Dr. R.S Kaushal for many helpful suggestions and critical reading of the manuscript.

REFERENCES
Burgan JR et al (1979).Solution of The Multi Dimensional Quantum Harmonic Oscillator With Time dependent Frequencies through Fourier, Hermite and Wigner Transforms.,Physics.**74 A**(1,2)11.

ErmakovVP(1880). Second order differential equation,Conditions of complete integrability.University Izv. KievSeries III **9** 1.

Hall LS (1983). Theory of exact and approximate configurational Invariants Physica **8D** 90.

Hietarinta (1987). Direct Method for the search of the second invariant Physics Reports**147**(2)87.

Kaushal RS (1998). Classical and Quantum Mechanics of Non-central Potentials**: "** A Survey of two dimensional systems" (Jointly published by Narosa Publishing House and Springer Verlag,Heidelberg).

Kaushal RS, **Gupta Shalini (2001).**Construction of exact dynamical Invariants in coupledOscillator problem *Journal of PhysicsA;Mathematical and General*. **34** 9879.

Kaushal RS,**Korsch HJ(1981).**Dynamical Noether`s invariants for time dependent nonlinear system *Journal of Mathematical Physics* **22**(9) 1904.

Kaushal RS and **Mishra SC(1993).**Dynamical algebraic approach and invariants for time dependent Hamiltonian systems in two dimensions *Journal of Mathematical Physics* **34** 5843.

Kaushal RS, Parashar D, Gupta Shalini and **Mishra SC(1997).**Construction of exact invariants for classical Dynamical Systems in three dimensions Annals of Physics 2**59** (2) 233.

KorschHJ (1979). Dynamical Invariants and time dependent Harmonic Systems Physics Letters **74A** (5) 274.

Lewis HR(1968). Class of Exact Invariants for Classical and Quantum time dependent Harmonic Oscillators *Journal of Mathematical Physics*. **9**(11)1976.

LutzkyM(1978). Noether`s Theorem and the time dependent harmonic oscillator Physics Letters **A6**8(1) 3.

Ray JR (1979). Cosmological particle creation PhysicalReview **D20**(10) 2032.

RayJR(1980). Noether's theorem and exact Invariants for time dependent systems Journal of PhysicsA Mathematical and general **13** 1969.

Ray JR and **Reid JL (1982).**Exact solutions to the time dependent Schrödinger equation. Physical Review A26 (2)1042; **Ray JR** and **Reid JL** (1971). More exact invariants for the time dependent Harmonic oscillator Physical Review **A71**(4) 317; **Ray JR** and **Reid JL**(1979).Exact time dependent invariants for N Dimensional systems Physical Review **A74**(1,2) 23

Tabor M(1989). Chaos and Integrability in Non-Linear Dynamics(John Wiley and Sons New York).

Takayama K(1982). Dynamical Invariant For Forced Time Dependent Harmonic Oscillator Physics letters **88** A(2) 55.

Whittaker ET(1927). Analytical Dynamics(University Press Cambridge).

Xu XW(2000).Exactly solving a two dimensional time dependent coupled quantum Oscillator Journal of Physics A; Mathematical and General **33** 2447.