ON THE NON-LINEAR SCHRÖDINGER TYPE EQUATION

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ABSTRACT

The 3D non-linear Schrödinger type Equation with the appropriate initial-boundary conditions is considered. By introducing a new functions the equation is reduced to the system of partial differential equations. In some cases the effective solutions are obtained. The approximate solution is constructed by means of explicit finite-difference schemes.

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Key Words: Non-Linear Schrödinger Equation, System Of Partial Differential Equations, Finite-Difference Schemes.

INTRODUCTION

Schrödinger Equation describes various physical phenomena. Linear Schrödinger Equation Describes quantum processes and in some special cases the effective solutions are obtained (see for example Auletta et. al(2009), Landau and Lifshitz (1977), Simon (2000)).

Non-linear Schrodinger type equation describes electron plasmic waves, waves in ultraconducted fluid, electromagnetic ion cyclotron waves with sufficiently large amplitude, waves in cosmic gases, etc., Alexandrov et. al (1984), Anderson and Hamilton (1993), Gurevich and Shvartsburg (1973), Hasegava and Matsumoto (2002), Lions (1969), Summers et. al (1998), Tsintsadze et. al (2010), Sulem (1999), Simon (2000), Whitham (1974), Zakharov and Shabat (1972).

Numerous works are devoted to the spectral properties of Schrödinger operator: Cazenave and Weissler (1989), Ginibre and Velo (1979), Kwong (1989), Weinstein (1989), Simon (1997), Sulem (1999), Laptev et. al (2005), Safronov (2004),. Blow –up solutions in time of critical non-linear Schrödinger equation were studied by Merle (1990), Merle and Raphael (2005), Perelman (2001), Bourgain and Wang (1998).

Here we consider non-linear Schrödinger equation in R^3 and some parallelepiped. By introducing a new functions the equation is reduced to the system of partial differential equations, which is more convenient

for investigation. The bounded solutions in R^3 are constructed. In 1D the solutions with singularities are fond at some interval. In 2D parallelepiped the approximate solution is constructed by means of explicit finite-difference schemes.

SETTING OF THE PROBLEM

In 3D space let us choose the coordinate system Oxyz and consider some area G_0 . We admit that

 $G_0 = R^3$ or G_0 is a parallelepiped $G_0 = \{0 < x < a_0, 0 < y < b_0, 0 < z < c_0\}$, where a_0, b_0, c_0 are the definite constants.

In the area $Q_T = G_0 \times \{0 < t < T\}$, we consider the following Schrödinger's type Equation

$$i\frac{\partial\Psi}{\partial t} + \Delta\Psi + \lambda |\Psi^2|\Psi = 0, \qquad (1)$$

with the initial condition

$$\Psi\Big|_{t=0} = \Psi_0(x, y, z) = U_0 + iV_0;$$
⁽²⁾

and with the boundary condition (if G_0 is bounded)

 $\left\|\Psi\right\|_{\partial G_0}=C_0,$

where ψ is a wave function, $\psi = U + iV$, λ is some parameter, Ψ_0 is the given continuous functions,

 C_0 is a constant, ∂G_0 is a boundary of G_0 .

The Cauchy problem (1), (2) is locally well-posed (Lions (1969), Cazenave and Weissler (1989).Ginibre and Velo (1979)).

CONSTRUCTION OF THE SOLUTIONS

The equation (1) is equivalent to the following system of partial differential equations

$$\begin{cases} \frac{\partial U}{\partial t} = -\Delta V - \lambda V \left(U^2 + V^2 \right), \\ \frac{\partial V}{\partial t} = \Delta U + \lambda U \left(U^2 + V^2 \right), \end{cases}$$
(3)

Let us introduce the notations

 $U = r \cos \varphi; \quad V = r \sin \varphi,$

Taking into the account (4) the system (3) becomes

$$\frac{\partial r}{\partial t}\cos\varphi - \sin\varphi \frac{\partial\varphi}{\partial t} = -\sin\varphi \,\Delta r - 2\cos\varphi \left(\frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\varphi}{\partial z}\right)$$
(5)
$$-r \left\{ -\sin\varphi \left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right) + \cos\varphi \,\Delta\varphi \right\} - \lambda r^3 \sin\varphi;$$
(5)
$$\frac{\partial r}{\partial t}\sin\varphi + r\cos\varphi \frac{\partial\varphi}{\partial t} = \cos\varphi \,\Delta r - 2\sin\varphi \left(\frac{\partial r}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial\varphi}{\partial z}\right)$$
(6)
$$-r \left\{ -\cos\varphi \left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right) + \sin\varphi \,\Delta\varphi \right\}$$
(6)

After simple transformations from (5), (6) we obtain

$$-r\frac{\partial\varphi}{\partial t} = -\Delta r + r\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right\} - \lambda r^3;$$
(7)

$$\frac{\partial r}{\partial t} = -2\left(\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial z}\right) - r\Delta\varphi;$$
(8)

Let u rewrite (7), (8) in the form

$$\frac{\Delta r + \lambda r^3}{r} = \frac{\partial \varphi}{\partial t} + \left\{ \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right\} ; \qquad (9)$$

$$\Delta \varphi = -\frac{\partial}{\partial t} \ln r - \left\{ \frac{\partial}{\partial x} (\ln r) \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial y} (\ln r) \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial z} (\ln r) \frac{\partial \varphi}{\partial z} \right\}; \tag{10}$$

Let us consider some particular cases:

1. Suppose

$$r = r_0(x, y, z)e^{\gamma}; \quad \varphi = \varphi_0(x, y, z) + D_1 e^{\gamma_0 t} + A_0 t, \tag{11}$$

where C_0 ; A_0 ; D_1 ; γ ; γ_0 are some constants, r_0 ; φ_0 are the functions to be determined. Putting (11) into (9), (10) we obtain

$$\frac{\Delta r_0}{r_0} + \lambda e^{2\gamma t} r_0^2 = D_1 \gamma_0 e^{\gamma_0 t} + A_0 + \left\{ \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \varphi_0}{\partial y} \right)^2 + \left(\frac{\partial \varphi_0}{\partial z} \right)^2 \right\} ; \qquad (12)$$

$$\Delta \varphi_0 = -\alpha - 2 \left\{ \frac{\partial}{\partial x} (\ln r_0) \frac{\partial \varphi_0}{\partial x} + \frac{\partial}{\partial y} (\ln r_0) \frac{\partial \varphi_0}{\partial y} + \frac{\partial}{\partial z} (\ln r_0) \frac{\partial \varphi_0}{\partial z} \right\};$$
(13)

2. In case of $\gamma = 0$; $r = r_0 = const$ from (12), (13) we obtain

$$\lambda r_0^2 = \frac{\partial \varphi}{\partial t} + \left\{ \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \varphi_0}{\partial y} \right)^2 + \left(\frac{\partial \varphi_0}{\partial z} \right)^2 \right\}$$
(14)

and effective solution of (14) will be given by (Courant and Hilbert (1989)) $\varphi = A_1 x + B_1 y + C_1 z + A_0 t + D_1$, (15) where

$$A_1^2 + B_1^2 + C_1^2 + A_0 = \lambda r_0^2;$$
(16)

Consequently, the solution of the system (3) is of the form

$$U = r_0 \cos(A_1 x + B_1 y + C_1 z + A_0 t + D_1),$$
(17)

$$V = r_0 \sin(A_1 x + B_1 y + C_1 z + A_0 t + D_1),$$
(18)

where A_1, B_1, C_1A_0 are the constants connected with the condition (16).

The solutins (16), (17) are bounded in R^3 and satisfies the following initial conditions $U = r_0 \cos(A_1 x + B_1 y + C_1 z + D_1); \quad t = 0,$

$$V = r_0 \sin(A_1 x + B_1 y + C_1 z + D_1); \quad t = 0.$$

It is obvious $U^2 + V^2 = r_0^2$. The graphs of (17) and (18) are plotted by means of the computer program Maple and are given in the figures Fig.1 and



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Fig 1: Profile of U **in the case** $\lambda = A_1 = B_1 = C_1 = A_0 = D_1 = 1; r_0 = 2; z = 1; t = 1.$

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Fig 2: Profile of *V* **in the case** $\lambda = A_1 = B_1 = C_1 = A_0 = D_1 = 1; r_0 = 2; z = 1; t = 1.$

3. Now, let us suppose

 $r = r_0 e^{\gamma t}; \quad \varphi = \varphi_0(x, y, z) + D_1 e^{\gamma_0 t} + A_0 t, \ r = r_0 = const$, than the system (12), (13) becomes

$$\lambda e^{2\gamma t} r_0^2 = D_1 \gamma_0 e^{\gamma_0 t} + A_0 + \left\{ \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \varphi_0}{\partial y} \right)^2 + \left(\frac{\partial \varphi_0}{\partial z} \right)^2 \right\} ; \qquad (19)$$

$$\Delta \varphi_0 = -\alpha ; \qquad (20)$$

The solution of this system exists only in the case $r_0^2 = D_1 \gamma_0$; $\gamma_0 = \gamma = 0$; $A_0 < 0$; and is given by (15), (16).

In case of $r_0^2 = D_1 \gamma_0; \gamma_0 = 2\alpha; \alpha = -2; A_0 < 0; \varphi = A_1 x^2 + A_1 y^2 + A_1 z^2 + A_0 t + D_1$, the system (19), (20) is satisfied only at the sphere $A_1(x^2 + y^2 + z^2) = -A_0$.

4. In case of $\varphi = A_0 t + D_1$, r does not depend on time and (12) implies $\Delta r + \lambda r^3 = A_0 r$. (21) Here $r; r \ge 0$ and the unique solution of this type exists Kwong (1989).

In the case, when r depends only on one variable x, the equation (21) could be reduced to the following differential equation

$$p^{2} = A_{0}r^{2} - \frac{\lambda}{2}r^{4} - 2B_{0} + B_{1}^{2}, \qquad (22)$$

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where
$$p = \frac{\partial r}{\partial x}; B_0 = \frac{A_0}{2}r_0^2 - \frac{\lambda}{4}r_0^4; B_1 = \frac{p^2(0)}{2}; r_0 = r(0).$$

The solution of the equation (22) will be given in the implicit form by the formula

$$x = -k \frac{\sqrt{2}}{\sqrt{-\lambda}} \int_0^{r/r_1} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} - r_0,$$
(23)

where

$$k = \frac{1}{r_2}; r_2 = \sqrt{D_0 - \frac{A_0}{\lambda}}; \quad r_1 = \sqrt{D_0 + \frac{A_0}{\lambda}};$$
$$D_0 = \frac{1}{|\lambda|} \sqrt{A_0^2 - \lambda(4B_0 - 2B_1^2)}; \quad A_0^2 - \lambda(4B_0 - 2B_1^2) \ge 0$$

(23) is the elliptic integral with the modulus k. Inversion of this integral is the Jakobi sinus and is given by Courant and Hurvitz (1929)

$$r = r_1 sn(-k_1 x + r_0); k_1 = \frac{\sqrt{-\lambda}}{k\sqrt{2}};$$
(24)

As we seak for real solution, we will consider the following cases:

1.
$$\lambda \ge 0$$
; $0 \le -k_1 x + r_0 \le K'$; $K' = \int_{-1}^{\frac{1}{k}} \frac{dt}{\sqrt{(t^2 - 1)(1 - k^2 t^2)}}$

Solutions of this type has singularity at the point K';

2.
$$\lambda \le 0$$
; $0 \le -k_1 x + r_0 \le K$ $K = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$

In the case of 2 variables the numerical solution of the equation (21) will be given by the explicit finitedifference schemes. In this case we consider the equation (15) at some rectangular area G_0 with the boundary conditions

$$r|_{\partial G_0} = r_0 = const;$$

where ∂G_0 is a boundary of the area $G_0 = \{0 < x < a_0, 0 < y < b_0\}$. Let us introduce the notation $u = r - r_0$ and rewrite (21) in the form

$$\Delta u = f_0 u^3 + f_1 u^2 + f_2 u + f_3, \ u\Big|_{\partial G_0} = 0,$$
(25)

where $f_0 = -\lambda$; $f_1 = 3\lambda r_0$; $f_2 = A_0 - 3\lambda$; $f_3 = \lambda r_0^3 - A_0 r_0$.

The area of integration G_0 we divide by the planes $x_i = ih_1$; $y_j = jh_2$; i = 0, 1, 2, ..., M; j = 0, 1, ..., N; into cells, where $h_1 = a_0 / M$; $h_2 = b_0 / N$;

Consequently, for the area G_0 we introduce the following grids

$$\boldsymbol{\varpi}_{h} = \{x_{i} = ih_{1}; y_{j} = jh_{2}; i = 0, 1, 2, ..., M; j = 0, 1, ..., N; \}$$

For net functions and their difference derivatives we introduce the following notations

$$u_{x} = \frac{1}{h_{1}} (u(x+h_{1}, y) - u(x, y)),$$

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$$u_{y} = \frac{1}{h_{2}}(u(x, y + h_{2}) - u(x, y)),$$

$$u_{\overline{x}} = \frac{1}{h_{1}}(u(x, y) - u(x - h_{1}, y)),$$

$$u_{\overline{y}} = \frac{1}{h_{2}}(u(x, y) - u(x, y - h_{2})),$$

$$\Delta_{1}u = \frac{1}{2}(u_{x} + u_{\overline{x}}), \ \Delta_{2}u = \frac{1}{2}(u_{y} + u_{\overline{y}}),$$

$$\Delta_{11}u=u_{xx}, \quad \Delta_{22}u=u_{yy},$$

For the problem (25) we have constructed the following finite-difference schemes

$$\begin{aligned} &-\sigma\tau^{2}\Delta_{11}u_{i\bar{i}} = u_{x_{1}\bar{x}_{1}} + u_{x_{2}\bar{x}_{2}} + f , \qquad (26) \\ &f = f_{0}u^{3} + f_{1}u^{2} + f_{2}u + f_{3}, \\ &-\sigma\tau^{2}\Delta_{11}\frac{u^{k+1} - 2u^{k-1}}{\tau^{2}} = u_{x_{2}\bar{x}_{1}}^{k} + u_{x_{2}\bar{x}_{2}}^{k} + (f_{0}u^{3} + f_{1}u^{2} + f_{2}u + f_{3})_{ij}^{k}, \qquad (27) \\ &- \left\{\frac{u^{k+1}_{i-1,j} - 2u^{k+1}_{ij} + u^{k+1}_{i-1,j}}{h_{1}^{2}} - 2\frac{u^{k}_{i-1,j} - 2u^{k}_{ij} + u^{k}_{i+1,j}}{h_{1}^{2}} + \frac{u^{k-1}_{i-1,j} - 2u^{k-1}_{ij} + u^{k-1}_{i+1,j}}{h_{1}^{2}}\right\} = \phi_{ij}, \\ &- \frac{\sigma}{h_{1}^{2}}\left(u^{k+1}_{i-1,j} + u^{k+1}_{i+1,j}\right) + 2\frac{\sigma}{h_{1}^{2}}u^{k+1}_{ij} = F_{ij} , \qquad (28) \\ F_{ij} &= -\frac{2\sigma}{h_{1}^{2}}\left[u^{k}_{i-1,j} - 2u^{k}_{ij} + u^{k}_{i+1,j}\right] + \frac{1}{h_{1}^{2}}\left(u^{k}_{i-1,j} - 2u^{k}_{ij} + u^{k}_{i+1,j}\right) + \frac{\sigma}{h_{1}^{2}}\left(u^{k-1}_{i-1,j} - 2u^{k-1}_{ij} + u^{k-1}_{i+1,j}\right) \\ &+ \frac{1}{h_{2}^{2}}\left(u^{k}_{i,j-1} - 2u^{k}_{ij} + u^{k}_{i,j+1}\right) + \left\{f_{0}u^{3}_{ij} + f_{1}u^{2}_{ij} + f_{2}u_{ij} + f_{3}\right\}_{ij}^{k}. \qquad (29)
\end{aligned}$$

The schemes (26), (27), (28), (29) are absolutely stable and convergent. The accuracy of this schemes is $O(h^2)$ and is proved as in Komurjishvili (2007).

CONCLUSION

1. In 3D there exists the bounded solution of the problem (1),(2), which is given by the formulas (16),(17), (18).

2. It is clear from (9), (10), that if φ depends only on time, then r does not depend on time.

3. In 1D case solutions of (1),(2) of the type $\Psi = re^{i(A_0t+D)}$ exist only at some interval :

I) For $\lambda \ge 0$; $0 \le -k_1 x + r_0 \le K'$; and has singularity at the point *K*';

II) For $\lambda \leq 0$; $0 \leq -k_1 x + r_0 \leq K$,

where $k_1; r_0; K; K'$ are definite constants.

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