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ON THE GENERALIZED HEAT EQUATION

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ABSTRACT
In this paper we have considered the general heat equation to study heat transform for a suitable f with the help of source solution. The inversion of the integral equation (3.4), and then deduced the inversion of the heat transform (3.1). Finally operational calculus is developed and some special cases are studied.

Key Words: *Heat Equation, Green Function, Heat Transform, Hankel Type Transform, Laplace Transform*

1. INTRODUCTION: The general heat equation is defined as

$$\frac{\partial^2 u}{\partial x^2} + \frac{4a}{x}\frac{\partial u}{\partial x} - \frac{d^2}{x^2}\, u = \frac{\partial u}{\partial x} \tag{1.1}$$

Or

$$\Delta_x u = \frac{\partial u}{\partial t} \ , \Delta_x \equiv \frac{\partial^2}{\partial x^2} + \frac{4a}{x}\frac{\partial}{\partial x} - \frac{d^2}{x^2}$$

where $2a$ is a fixed positive number and d is a fixed number. If $2a = d = 0$, then (1.1) reduces to the ordinary heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad ,$$

where $u(x,t)$ is regarded as the temperature at a point x at time t, in an infinite insulated rod extended along the x −axis in the xt −plane . If we set $a = \frac{1}{4}$ then (1.1) becomes

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial F}{\partial t} \quad ,$$

the heat equation in two dimensions, where the solutions are of the type

$$F(x,y,t) = u(r,t) sin d\theta$$

in polar coordinates ; and represents the temperature in a plane sector of angle π/d.

Further, if we put $a = \frac{1}{2}$ and $d^2 = n(n+1)$, then (1.1), yields the heat equation in three dimensions

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \frac{\partial F}{\partial t}$$

where the solutions are of the form

$$F(x, y, z; t) = u(r, t)\, P_n\,(\cos\phi)$$

in spherical coordinates representing the temperature in a cone of angle ϕ. Here $P_n(Z)$ are the Legendre polynomials. Consequently the heat equation (1.1) can be regarded as representing a general situation for the flow of heat.

The object of this paper is to study the analytic consequences of the general heat equation. We shall devote our main effort towards establishing some properties of the source solution and an algorithm for the inversion of the heat transform. The case $d = 0$ has been dealt with thoroughly in *Haimo and Cholewinski*[1966].

2. THE SOURCE SOLUTION:

Consider the temperature at $x = p,$ as instantaneously enormous at $t = 0^+$ but leveling off rapidly. Thus there is a source at $x = p$; and the temperature function is now defined as the source solution. To find the source solution $u(p, x, t)$ of (1.1), we consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{4a}{x}\frac{\partial u}{\partial x} - \frac{d^2}{x^2}u = \frac{\partial u}{\partial t} - \delta(x-p)\,\delta(t) \qquad (2.1)$$

where δ is the Dirac delta function. If

$$\overline{u} = \int_0^\infty u(x, t)\, e^{-st}\, dt,$$

then (2.1) gives,

$$\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{4a}{x}\frac{\partial \overline{u}}{\partial x} - \frac{d^2}{x^2}\,\overline{u} = s\overline{u} + \delta(x-p).$$

The solution is

$$\overline{u} = \begin{cases} p^{3a+b}\, x^{-(a-b)}\, K_{\alpha-\beta}\left(s^{\alpha+\beta}p\right) I_{\alpha-\beta}\left(s^{\alpha+\beta}x\right), & x < p \\ p^{3a+b}\, x^{-(a-b)} I_{\alpha-\beta}\left(s^{\alpha+\beta}p\right) K_{\alpha-\beta}\left(s^{\alpha+\beta}x\right), & x > p \end{cases}$$

and by the inverse Laplace transform, *Erdelyi ETAL*[1954, p. 284],

$$U \equiv U(p, x, t) = p^{4a}\, G_{\alpha-\beta}\,(p, x; t), \qquad (2.2)$$

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where

$$G_{\alpha-\beta}\,(p,x;t) = \frac{1}{2t}\,(px)^{-(a-b)}\,e^{-(p^2+x^2)/4t} I_{\alpha-\beta}\,\left(\frac{px}{2t}\right) = G(p,x;t),$$

say, and

$$(\alpha-\beta)^2 = (a-b)^2 + d^2, \qquad (\alpha-\beta) > -1, \qquad t > 0.$$

We shall call the function U to be the source solution of the general heat equation (1.1) and for the simplicity we shall say that

$$U(p,x;t) \in H. \tag{2.3}$$

Further we shall discuss some of the more interesting properties of the source solution and in particular, the so-called Green's function $G(p,x;t)$. We note that

$$G(p,x;t) = (px)^{-(a-b)} \int_0^\infty u\,e^{-tu^2}\,J_{\alpha-\beta}(pu)\,J_{\alpha-\beta}(xu)\,du,$$

where $(\alpha-\beta) > -1$, $t > 0$, Erdelyi ET AL$[1954, p.51]$. As a direct result of the definition of the function $U(p,x;t)$, we have the following theorem.

Theorem 2.1: Let $U(p,x;t)$ be defined as above. Then

(i) $U(p,x;t) > 0, \quad p,\ x > 0$,

(ii) $U(\lambda p,\ \lambda x;\ \lambda^2 t) = U(p,\ x;t)$,

(iii)$x^{2(a-b)+1}\,U(p,x;t) = p^{2(a-b)+1}\,U(x,p;t)$.

Theorem 2.2: Let $U(p,x;t)$ be as defined above. Then

(i) $\int_0^\infty p^k\,U(p,x;t)\,dp = x^k$,

(ii)$|U(p,s;t)| \leq A(2t)^{-(\alpha+\beta)}|p|^{a-b+\frac{1}{2}}\,(\sigma^2+\tau^2)^{-a}\,e^{-((\sigma-p)^2+t^2)/4t}$, (2.4)

where $s = \sigma + i\tau$, $\sigma > 0$, $-\infty < \tau < \infty$,

(iii) $\left|\frac{\partial}{\partial s}\,U(p,s;t)\right| \leq B\,t^{-1}\,|p|^{a-b+\frac{1}{2}}\,(\sigma^2+\tau^2)^{-a}\,e^{-\{(\sigma-p)^2+\tau^2/4t}\,\left(\frac{b-a+\alpha-\beta}{\sqrt{\sigma^2+\tau^2}} + \frac{\sqrt{\sigma^2+\tau^2}}{2t} + \frac{p}{2t}\right)$, (2.5)

(iv) $\left|\frac{\partial}{\partial t}\,U(\xi,x;t)\right| \leq C\,t^{-3/2}\,\left(\frac{p}{x}\right)^{a-b+\frac{1}{2}}\,e^{-(p-x)^2/4t}\,\left(-(\alpha+3\beta) + \frac{(p-x)^2}{4t}\right)$. (2.6)

Proof: Conclusion (i) follows by direct computation. From the definition given in (2.2) and the asymptotic behavior

$I_{a-b+\frac{1}{2}}(z) \sim \left(1/\sqrt{2\pi z}\right)e^z$, we have

$$U(p,s;t) = \frac{p^{2(a-b)+1}}{2t} (ps)^{-(a-b)} e^{-(p^2+s^2)/4t} I_{\alpha-\beta}\left(\frac{ps}{2t}\right),$$

and it follows that

$$|U(p,s;t)| \le A(2t)^{-1/2} |p|^{a-b+\frac{1}{2}} (\sigma^2 + \tau^2)^{-a} e^{-\{(\sigma-p)^2+\tau^2/4t.}$$

Also,

$$\frac{\partial}{\partial s} U(p,s;t) = \frac{\partial}{\partial s}\left\{\frac{p^{2(a-b)+1}}{2t} (ps)^{-(a-b)} e^{-(p^2+s^2)/4t} I_{\alpha-\beta}\left(\frac{ps}{2t}\right)\right\}$$

$$= \frac{p^{3a+b+\alpha-\beta}}{(2t)^{3\alpha+\beta}} e^{-p^2/4t} \frac{\partial}{\partial s}\left\{s^{-(a-b)+\alpha-\beta} e^{-s^2/4t} \left(\frac{ps}{2t}\right)^{-(\alpha-\beta)} I_{\alpha-\beta}\left(\frac{ps}{2t}\right)\right\}$$

$$= \frac{1}{2t} p^{3a+b} e^{-(p^2+s^2)/4t}\left\{(\alpha-\beta-a+b)\, s^{-(3a+b)} I_{\alpha-\beta}\left(\frac{ps}{2t}\right) -\right.$$

sa+3b2t Iα−β ps2t+ p2t s−a−b I3α+β ps2t

$$\sim \frac{1}{(2t)^{\frac{1}{2}}}\left(\frac{p}{s}\right)^{a-b+\frac{1}{2}} e^{-(p-s)/4t}\left[(\alpha-\beta-a+b)\frac{1}{s} + \frac{s}{2t} + \frac{p}{2t}\right]$$

Thus

$$\left|\frac{\partial}{\partial s} U(p,s;t)\right| \le \frac{|p|^{a-b+\frac{1}{2}}}{(2t)^{\frac{1}{2}}} (\sigma^2+\tau^2)^{-a} e^{-((\sigma-\rho)^2+\tau^2)/4t}\left[\frac{\alpha-\beta-a+b}{\sqrt{\sigma^2+t^2}} + \frac{\sqrt{\sigma^2+t^2}}{2t} + \frac{p}{2t}\right],$$

proving the assertion (iii). To prove the assertion (iv), by direct computation, we have

$$\frac{\partial}{\partial t} U(p,x;t) = \frac{1}{4t^2} U(p,x;t)\,(x^2+p^2-4t+4(\alpha-\beta)t^2) - \frac{px}{2t^2} U_{-(\alpha+3\beta)}(p,x;t)$$

hence

$$\left|\frac{\partial}{\partial t} U(p,x;t)\right| \le\subset t^{-\frac{3}{2}}\left(\frac{p}{x}\right)^{a-b+\frac{1}{2}} e^{-(p-x)^2/4t}\left\{-(\alpha+3\beta) + \frac{(p-x)^2}{4t}\right\},$$

as required

Theorem 2.3: If $0 \le x < \infty$, $0 \le y < \infty$ and $0 < t_1 < t_2$, then

(i) $\int_0^\infty U(p,x;t)\, G(p,y;t)\, dp = G(x,y;\, t_1+t_2)$,

(ii) $\int_0^\infty U(p,ix;\, t_1)\, G(ip,\, y;\, t_2) = (-1)^k\, G(x,y;\, t_2-t_1)$

where

$$k = \alpha-\beta-(a-b)\ and\ (\alpha-\beta) > -1.$$

Proof: By using the estimates derived in the last theorem it is easy to show that the integrals in the assertions (i) and (ii) above exist. Now by direct evaluation $Erdelyi\ ET\ AL$[1954, $p.$197].

$$\int_0^\infty U(p,x;\ t_1)\ G(p,y;\ t_2)dp$$

$$= \frac{(xy)^{-(a-b)}}{4\,t_1\,t_2}\ e^{-(x^2/t_1+y^2/t_2)/4} \int_0^\infty I_{\alpha-\beta}\left(\frac{px}{2t_1}\right) I_{\alpha-\beta}\left(\frac{py}{2t_2}\right) e^{-p^2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)/4}\ pdp$$

$$= \frac{1}{2(t_1+t_2)}\ (xy)^{-(a-b)}\ e^{-(x^2+y^2)/4\,(t_1+t_2)} I_{\alpha-\beta}\left(\frac{xy}{2(t_1+t_2)}\right)$$

$$= G(x,y;\ t_1+t_2)$$

as required. Also, the assertion (ii) can similarly be established. Note that assertion (ii) can be considered as the inversion of the integral equation in (i).

3. THE HEAT TRANSFORM:

If we now consider the source solution U as the Kernel, then for a suitable f, its heat transform F is defined by

$$x^k\,F(x,t) = \int_0^\infty U(p,x;t)\,p^k f(p)dp \qquad (3.1)$$

where $k = \alpha-\beta-(a-b)$, $(\alpha-\beta)^2 = (a-b)^2 + d^2\ and\ (\alpha-\beta) > -1$.

Theorem 3.1: If $f(x)$ is bounded and continuous in $0 < x < \infty$, and has a heat transform $F(x,t)$, then $x^k\,F(x,t) \in H$, $t > 0$, where $k = \alpha-\beta-(a-b)$, $(\alpha-\beta) > -1$.

Proof: From (3.1) above,

$$|x^k\,F(x,t)| \le A\int_0^\infty U(p,x;t)\,p^k\,dp$$

$$< Bt^{-\frac{1}{2}}\,x^{-\left(a-b+\frac{1}{2}\right)} \int_0^\infty p^{a-b+k}\,e^{-(p-x)^2/4t}\,dp < \infty,$$

using the estimate (2.4), where $A = u.b.f(x)$, $0 < x < \infty$. Hence the integral defining the function F exists and is in fact absolutely convergent. Now

$$\Delta_x\,[x^k\,F(x,t)] = \Delta_x\left[\int_0^\infty U(p,x;t)\,p^k\,f(p)\,dp\right]$$

$$= \int_0^\infty \Delta_x\,[U(p,x;t)]\,p^k\,f(p)\,dp$$

$$= \int_0^\infty \frac{\partial}{\partial t} U(p,x;t)\, p^k f(p) dp$$

$$= \frac{\partial}{\partial t} \int_0^\infty U\,(p,x;t)\, p^k\ f(p)\, dp$$

$$= \frac{\partial}{\partial t}\,[x^k F(x,t)]\,,$$

proving that $x^k F(x,t)$ satisfies the heat equation (1.1), hence $x^k F(x,t) \in H$, provided one can justify interchanging the operators Δ_x and $\frac{\partial}{\partial t}$ with the integral sign. Now, using (2.6), we have

$$\int_0^\infty \Delta_x\,[U(p,x;t)]\, p^k f(p)\, dp = \int_0^\infty \left|\frac{\partial}{\partial t}\,[U(p,x;t)]\, p^k f(p)\right| dp$$

$$\leq A\,(x,t) \int_0^\infty p^{k+a-b+\frac{1}{2}}\ e^{-(p-x)^2/4t}\ |f(p)|\, dp$$

$$\leq B(x,t) \int_0^\infty p^{k+a-b+\frac{1}{2}}\, e^{-p^2/8t}\ |f(p)|\, dp\ < \infty,$$

for all $x > 0$ and $t > 0$, and making use of the fact that

$$(p-x)^2\ \geq\ \frac{1}{2}\, p^2 - x^2$$

hence

$$e^{-(p-x)^2/4t}\ \leq\ e^{-p^2/8t\, +\, x^2/4t}\ ,$$

giving us the desired justification.

Theorem 3.2: If $f(x)$ is bounded and continuous in $0 < x < \infty$, then

(i) $s^k F(s,t) = \int_0^\infty U(p,s;t)\, p^k f(p)\, dp$, (3.2)

$s = \sigma + i\tau$, $\sigma > 0$, $-\infty < \tau < \infty$, *and*

(ii) $s^k F(s,t)$ is analytic in the complex half plane $Re s > 0$.

Proof:

$$|s^k F(s,t)|\ \leq \int_0^\infty |U(p,s;t)\, p^k f(p)|\, dp$$

$$\leq A\,(\sigma,\tau,t)\int_0^\infty p^{k+a-b+\frac{1}{2}}\,e^{-(p-\sigma)^2/4t}\,|f(p)|dp\ <\ \infty,$$

due to (2.4). Hence the function $s^k\,F(s,t)$exists and is defined by a uniformly convergent integral for $\sigma > 0$ and t > 0.

To prove that $s^k\,F(s,t)$ is analytic in the complex half plane $\sigma > 0$, we need to show that

$$\int_0^\infty \frac{\partial}{\partial s}\,U(p,s;t)\,p^k\ f(p)\,dp$$

converges uniformly. Now, using (2.5) we have

$$\int_0^\infty \left|\frac{\partial}{\partial s}\,U(p,s;t)p^k\,f(p)\right|\,dp\ \leq B(\sigma,\tau,t)\int_0^\infty p^{a-b+\frac{1}{2}+k}\ e^{-(p-\sigma)^2/4t}\,|f(p)|\,dp$$

which converges uniformly for all $\sigma > 0$ and t > 0 as seen above.

Hence the theorem is proved.

Corollary8:$|s^kF(s,t)|\ \leq\ C\ (\sigma^2+\tau^2)^b\ e^{-(\sigma^2+\tau^2)/4t}$. (3.3)

Theorem 3.3: Let $F(x,t)$ be as defined above. Then

$$x^k\,F(x,t+t_1)\ =\ \int_0^\infty U(p,x;t)\,p^k\,F(p,t_1)\,dp \tag{3.4}$$

for a fixed $t_1 > 0$.

Proof: It is easy to see that the integral in (3.4) exists.

Now,

$$\begin{aligned}\int_0^\infty U(p,x;t)\,p^k\,F(p,t_1)\,dp &= \int_0^\infty U(p,x;t)dp\int_0^\infty U(y,p;t_1)\,y^k\,f(y)dy\\ &= \int_0^\infty y^k\,f(y)dy\int_0^\infty U(p,x;t)\,U(y,p;t_1)\,dp\\ &= \int_0^\infty y^{k+2(a-b)+1}\,f(y)\,dy\int_0^\infty U\,(p,x;t)\,G\,(y,p;t_1)\,dp\\ &= \int_0^\infty y^{k+2(a-b)+1}\,f(y)\,G(x,y;t+t_1)dy\end{aligned}$$

$$= \int_0^\infty U\,(y,x;t+t_1)\,y^k\,f(y)dy$$

$$= x^k\,F(x,t+t_1),$$

making use of the result of Theorem 2.3 and the equation (3.1).

The change of order of integration in the above analysis is justified due to absolute convergence. Thus equation (3.4) is proved.

Corollary: If $t_1 \to 0$, then (3.4) yields the heat transform (3.1), formally.

4. The inversion:

In this section we shall find the inversion of the integral equation (3.4), and then deduce the inversion of the heat transform (3.1).

Theorem 4.1: Let $F(x,t)$ be defined as above. If

$$x^k\,F(x,t+t_1) = \int_0^\infty U(p,x;t)\,p^k\,F(p,t_1)\,dp,$$

then

$$x^k\,F(x,t_1) = \int_0^\infty U(p,ix;t)\,\left(\frac{p}{t}\right)^k\,F(ip,t+t_1)dp \tag{4.1}$$

where $t,t_1 > 0,\ a-b > 0-\frac{1}{2}\quad,\ (\alpha-\beta) > -1\ and\ k = \alpha-\beta-(a-b).$

Proof: Now form (3.4),

$$(ix)^k\,F(ix,t+t_1) = \int_0^\infty U(p,ix;t)\,p^k\,F(p,t_1)\,dp,$$

or simplifying,

$$a^{a-b+k}F(ix,t+t_1) = \frac{1}{2t}\int_0^\infty p^{3a+b+k}\,e^{-(p^2-x^2)/4t}\,J_{\alpha-\beta}\left(\frac{px}{2t}\right)\,F(p,t_1)\,dp.$$

If we put $x = 2ty$, then above equation gives,

$$(2t)^{k+3a+b}\,y^{k+a-b+\frac{1}{2}}\,e^{-y^2t}\,F(i2ty,t+t_1)$$

$$= \int_0^\infty (py)^{\alpha+\beta}\,J_{\alpha-\beta}(py)\;p^{k+a-b+\frac{1}{2}}\,F(p,t_1)\,e^{-p^2/4t}\,e\;\;dp$$

which is the usual form of the Hankel type transform, and therefore on inverting gives

$$y^{k+a-b+\frac{1}{2}}\, e^{-y^2/4t}\, F(y,t_1)$$
$$= \int_0^\infty (py)^{\alpha+\beta} J_{\alpha-\beta}(py)\,(2t)^{k+3a+b}\, e^{-p^2 t}\, F(i2tp, t+t_1)\, p^{k+a-b+\frac{1}{2}}\, dp.$$

Again, let $v = 2tp$, and simplify to get

$$y^k F(y,t_1) = \int_0^\infty U(v, iy; t) \left(\frac{v}{t}\right)^k F(iv, t+t_1)\, dv.$$

Corollary 1: (3.4) can alternatively be written as (4.2)

$$y^k F(y,t_1) = \int_0^{i\infty} U(x,y;-t)\, x^k F(x, t+t_1)\, dx$$

Corollary 2: Let $t_1 \to 0$. Then

$$\lim_{t_1\to 0} F(x,t_1) = f(x)$$

and the pair of heat transforms (3.4) and (3.5) reduce to respectively,

$$x^k F(x,t) = \int_0^\infty U(p,x;t)\, p^k f(p)\, dp \tag{4.3}$$

and

$$x^k f(x) = \int_0^\infty U(p, ix; t) \left(\frac{p}{i}\right)^k F(ix,t)\, dp \tag{4.4}$$

formally.

5. Operational calculus:

By Taylor series expansion, we have

$$x^k F(x, t+t_1) = x^k \sum_{n=0}^\infty \frac{t^n}{n!}\left(\frac{\partial}{\partial u}\right)^n [F(x,u)]_{u=t_1}$$
$$= \sum_{n=0}^\infty \frac{t^n}{n!}\left(\frac{\partial}{\partial u}\right)^n [x^k F(x,u)]_{u=t_1}.$$

Since $x^k F(x,t) \in H$, that is

$$\Delta_x [x^k F(x,t)] = \frac{\partial}{\partial t}[x^k F(x,t)],\ \Delta_x \equiv \frac{\partial^2}{\partial x^2} + \frac{4a}{x}\frac{\partial}{\partial x} - \frac{d^2}{x^2}.$$

Thus from above, we have

$$x^k F(x, t + t_1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\Delta_x)^n [x^k F(x, t_1)]$$
$$= e^{t\,\Delta_x}{}^{[x^k F(x,t_1)]}.$$

The heat transform (3.4), can now be written symbolically as,

$$e^{t\Delta_x}[x^k F(x, t_1)] = x^k F(x, t + t_1), \tag{5.1}$$

where

$$e^{t\Delta_x} [\phi(x)] \equiv \int_0^{\infty} U(p, x; t) \phi(p) dp.$$

Therefore the inversion of (5.1) is then

$$x^k F(x, t_1) = e^{-t\Delta_x}[x^k F(x, t + t_1)], \tag{5.2}$$

where from (4.2),

$$e^{-t\Delta_x} [\psi(x)] \equiv \int_0^{i\infty} U(p, x; -t) \psi(p) dp.$$

In particular if we let $t_1 \to 0$, then formally, (5.1) and (5.2) reduce to the pair

$$e^{t\Delta_x}[x^k f(x)] = x^k F(x, t) \tag{5.3}$$

and

$$x^k f(x) = e^{-t\Delta_x} [x^k F(x, t)], \tag{5.4}$$

respectively, giving us a pair of heat transforms in operator form.

Example 1: Let

$$f(x) = x^{-(a-b)-k} I_{\alpha-\beta}(x).$$

Then its heat transform is $(see\ Waphare[2012])$,

$$F(x, t) = e^t x^{-(a-b)-k} I_{\alpha-\beta}(x) \quad , \qquad (\alpha - \beta) > -1.$$

Now,

$$e^{-t\Delta_x}[x^k F(x, t)] = e^{-t\Delta_x} \left[e^t x^{-(a-b)} I_{\alpha-\beta}(x)\right]$$
$$= e^t e^{-t\Delta_x} \left[x^{-(a-b)} I_{\alpha-\beta}(x)\right]$$
$$= e^t \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\Delta_x)^n \left[x^{-(a-b)} I_{\alpha-\beta}(x)\right].$$

But

$$\Delta_x \left[x^{-(a-b)}\, I_{\alpha-\beta}\,(x)\right] = x^{-(a-b)}\, I_{\alpha-\beta}(x),$$

hence

$$(\Delta_x)^n \left[x^{-(a-b)} I_{\alpha-\beta}(x)\right] = x^{-(a-b)}\, I_{\alpha-\beta}(x), \qquad n = 0,1,2, \ldots.$$

and

$$e^{-t\Delta_x}\,[x^k\, F(x,t)] = e^t \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\, x^{-(a-b)}\, I_{\alpha-\beta}(x)$$

where $$= x^{-(a-b)}\, I_{\alpha-\beta}(x) = x^k\, f(x) \text{ say,}$$

$$f(x) = x^{-(a-b)-k}\, I_{\alpha-\beta}(x)\,,$$

as required.

Example 2: Let

$$F(x,t) = 1.$$

Then

$$x^k\, f(x) = e^{-t\Delta_x}\,[x^k] = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\, (\Delta_x)^n\,[x^k] \quad .$$

Now,

$$\Delta_x\,[x^k] = \left(\frac{\partial^2}{\partial x^2} + \frac{4a}{x}\frac{\partial}{\partial x} - \frac{d^2}{x^2}\right)[x^k]$$

$$= (k(k-1) + 4ak - d^2)\, x^{k-2}$$

$$= 0,$$

since $k = \alpha - \beta - (a-b)$ *and* $(\alpha-\beta)^2 = (a-b)^2 + d^2$. Therefore

$$(\Delta_x)^n\,[x^k] = 0\;, \qquad n = 1,2,3, \ldots \ldots \ldots.$$

and

$$x^k\, f(x) = x^k\,,$$

so that

$$f(x) = 1.$$

Thus $f(x) = 1$ *and* $F(x,t) = 1$ gives us a pair of heat transforms which can be verified by evaluating the integrals (4.3) and (4.4).

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Research Article

6. Special Cases:

Let $d = 0$. The general heat equation (1.1) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{4a}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \quad ,$$

which has the solution, from (2.3) as

$$U(p, x; t) = p^{2(a-b)+1}\, G_{a-b}\,(p, x; t).$$

The heat transform (3.1) is then

$$F(x, t) = \int_0^\infty p^{4a}\, G_{a-b}\,(p, x; t)\, f(p)\, dp,$$

called the Poisson-Hankel transform, and its inversion is given by

$$f(x) = \int_0^\infty p^{4a}\, G_{a-b}\,(p, ix; t)\, F(ip, t) dp, \quad (See\ [5]).$$

If we let $d = a - b + \frac{1}{2} = 0$, $\alpha - \beta = -\frac{1}{2}$ then (1.1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad ,$$

the ordinary heat equation, whose source solution is

$$U(p, x; t) = G_{-\frac{1}{2}}(p, x; t) = -\frac{1}{\sqrt{\pi t}}\, e^{-(p^2+x^2)/4t} \cosh\left(\frac{px}{2t}\right).$$

Also form (4.3) and (4.4), we have

$$F(x, t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-(p^2+x^2)/4t} \cosh\left(\frac{px}{2t}\right) f(p)\, dp \quad ,$$

and

$$f(x) = \frac{1}{\sqrt{\pi t}} \int_0^{i\infty} e^{(p^2+x^2)/4t} \cosh\left(\frac{px}{2t}\right) F(p, t) dp.$$

Simbolically, the operator

$$\Delta_x = \frac{\partial^2}{\partial x^2} = D^2 \quad ,$$

then (5.3) and (5.4) yield

$$F(x, t) = e^{tD^2}[f(x)]$$

$$f(x) = e^{-tD^2}\,[F(x, t)]$$

which gives the Eddington solution of the ordinary heat equation ($see\ Sneddon$ [4, $p.$ 85]).

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