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Research Article

THE QUATERNIONS: AN INTRINSIC TREATMENT

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ABSTRACT

We define the quaternions $\mathbb{H}$ intrinsically, without recourse to considering units i, j, k= ij (as usually made), as "scalar and vector", $q = (u, \mathbf{x})$ in $R^4 = \mathbb{R} \oplus R^3$. Product $qq' = (uu' - \mathbf{x}\cdot\mathbf{x}', u\mathbf{x}' + u'\mathbf{x} + \mathbf{x}\wedge\mathbf{x}')$, conjugate $\bar{q}$, norm $\mathcal{N}(q)$ and inverse are found likewise, with the help of a definite quadratic form $Q(\mathbf{x})$, which allows to define norm and vector product. From any oriented orthonormal base e_i (i=1, 2, 3) we recover the usual units i, j, k; the complex algebra © is recovered when in the product qq' the two vectors $\mathbf{x}$, $\mathbf{x}'$ are parallel. Also, the quadratic equation $q^2 - (\mathrm{Tr}\, q)\cdot q + \mathrm{Det}(q) = 0$ holds, with solutions q' given by conjugation $\bar{q}$ and $g\cdot q$, where $g \in SO(3) = \mathrm{Aut}(\mathbb{H})$.

1. Traditional quaternions. W.R. Hamilton (1842) defined the quaternions $\mathbb{H}$ as a "field" in four real dimensions, but not commutative. He generalized the complex numbers $z = x+iy$, with $i^2 = -1$, to

$$q = u + xi + yj + zk = u + \dots \equiv \mathrm{Re}\, q + \mathrm{Im}\, q \qquad (1\text{-}1)$$

where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, and cyclically $jk = i$ and $ki = j$. In the usual way one defines the product qq', the *conjugate* $\bar{q}$ as $= u - xi - yj - zk$, the *norm* $\mathcal{N}(q) := \bar{q}\, q$ as $u^2 + (x^2 + y^2 + z^2)$, which is real ≥ 0, and the *inverse* as $q^{-1} = \bar{q} / \mathcal{N}(q)$. Conjugation is antiautomophism, meaning $\overline{(qq')} = \bar{q}'\, \bar{q}$, but any 3D rotation is an automorphism. In fact, for α an automorphism $\alpha(q) = \alpha(\mathrm{Re}\, q + \mathrm{Im}\, q) = \mathrm{Re}\, q + \alpha(\mathrm{Im}\, q)$; $\mathcal{N}(\mathrm{Im}\, q)$ is real, hence preserved under α, so one concludes

$$\mathrm{Aut}(\mathbb{H}) = SO(3) \qquad (1\text{-}2)$$

It is not the whole of O(3) because one must maintain orientation (ij= k); α is orthogonal because it maintains the 2-sphere S^2 of unit *imaginary* quaternions; also we used the fact that the field R has no automorphisms ≠ Id.

In this note we want to reproduce all algebraic results about quaternions without referring to three preselected units i, j, k; that is, we are to exhibit the *intrinsic nature* of the quaternion skew field; (the word *skew field* just means a non-commutative "field").

2. Intrinsic treatment. Consider, in R^4, the 1+3 split, $R^4 = \mathbb{R} + R^3$, where $V = R^3$ is the usual 3-dim real vector space. Choose in it a definite quadratic form Q: that is, chose a bilinear mapping f: $R^3 \times R^3 \to \mathbb{R}$; $f(\mathbf{x}, \mathbf{x}') := \mathbf{x}\cdot\mathbf{x}'$, which is i) *symmetric* ii) *regular* (nondegenerate: the associated map $\tilde{f}$: V in the dual V* is isomorphism; or: in any base, which converts f into a matrix F, is det F ≠0), and iii) *definite*, meaning $Q(\mathbf{x}) \equiv \mathbf{x}^2 > 0$ for $\mathbf{x} \neq$

0. It is well known that there is a *unique* Q up to equivalences, satisfying these three conditions (Sylvester).

The calculations now are straightforward. Define, in |R ⊕ R^3 the pair

$$q := (u, \mathbf{x}) \qquad (2\text{-}1)$$

(where u lies in R and $\mathbf{x}$ is any 3-vector in $V = R^3$) as a *quaternion*.

We have then a *skew* field, with the usual definitions $[q' := (u', \mathbf{x}')]$

$$q + q' = (u + u', \mathbf{x} + \mathbf{x}').- \qquad rq = (ru, r\mathbf{x}) \quad \text{(for any real r).}$$
$$qq' = [(uu' - \mathbf{x}\cdot\mathbf{x}'),\ u\mathbf{x}' + u'\mathbf{x} + \mathbf{x}\wedge\mathbf{x}'] \qquad (2\text{-}2)$$

The vector product $\wedge$ is well defined: if $\mathbf{x}$ and $\mathbf{x}'$ are parallel, the vector product is defined as zero. In 3 dimensions, for the biplane $< \mathbf{x}, \mathbf{x}' >$ there is a *unique* orthogonal direction; $\mathbf{x} \wedge \mathbf{x}'$ lies in this $\perp$ direction, and $\mathbf{x}\wedge\mathbf{x}'$ has modulus $|\mathbf{x}| \cdot |\mathbf{x}'|\ |\sin(\varphi)|$, where $Q(\mathbf{x}) := \mathbf{x}^2$ and $|\mathbf{x}| = +\sqrt{Q(\mathbf{x})}$. The angle φ, $0 \le \varphi \le 2\pi$ is measured from $\mathbf{x}$ to $\mathbf{x}'$. The rest of calculation (conjugate, norm and inverse) proceed as above:

$$\bar{q} = (u, -\mathbf{x}); \quad \bar{q}\, q = u^2 + \mathbf{x}^2 := \mathcal{N}(q) \ge 0; \quad q^{-1} = \bar{q} / \mathcal{N}(q) \ \ (q \ne 0) \qquad (2\text{-}3)$$

Notice $\mathcal{N}(q)$ is calculated from the above product qq'. If we define any orthoframe $\varepsilon = \{e_1, e_2, e_3\}$ with $e_1 \wedge e_2 = e_3$ cyclic etc., we recover the original Hamilton formulation with $i^2 = -1$, $ij = k$, etc., via the correspondence $e_1 = i$, $e_2 = j$.

We recover the Complex field © whenever the two vectors in q, q′ are parallel:

$$(u, \mathbf{x})(u', r\mathbf{x}) = [u u' - r\, \mathbf{x}\cdot\mathbf{x},\ \mathbf{x}(u' + ru) + r\, \mathbf{x}\wedge\mathbf{x}] = (uu' - r\mathbf{x}^2,\ \lambda\mathbf{x}) \qquad (2\text{-}4)$$

with $\lambda = u' + ru \in$ |R. Defining $i = \mathbf{x}^\wedge$ as a unit vector along $\mathbf{x}$, we recover the usual complex product, $(x + iy)\cdot(x' + iy')$, when $y = |\mathbf{x}|$ and $y' = r|\mathbf{x}|$ for |H.

To complete the study, define $Tr(q) := \bar{q} + q = 2u$ and $Det(q) := \mathcal{N}(q) = \bar{q}\, q$. Then the *quadratic* equation follows:

$$q^2 - (Tr(q))\cdot q + Det(q) \equiv qq - (\bar{q} + q)\cdot q + \bar{q}\, q \equiv 0 \qquad (2\text{-}5)$$

As an equation on q, we have the solutions $q = q'$ where

$$q' = (2u \pm \sqrt{-4\mathbf{x}^2})/2 = (u, \pm \mathbf{x}') \qquad (2\text{-}6)$$

That is, together with q, we have the conjugate $\bar{q}$ and any other q′ with the same real part and norm, $\mathbf{x}'^2 = \mathbf{x}^2$. It is nice that the equation (2-5) has as solutions the whole set AntiAuto(|H) $\cdot q$, if q is the original quaternion. (In the complex case one just obtains z and $\bar{z}$).

3. Comparison between division algebras. The reals |R, the complex ©, the quaternions |H and the octonions O (Graves, 1842) are the four division algebras over the real field |R (Baez, 2011). If we define in |R conjugation by the identity, we can play the same game as before in the four cases, for products, norm and inverses. We just notice some subtle differences in the quadratic equation:

R) For the *reals*, solutions of the quadratic equation $r^2 - (2r)\cdot r + r^2 = 0$ is $r' = r \pm 0$: this is consequence of the fact that the field R has *no* automorphisms, Aut(|R) = I.

C) For the *complex* case, (as said), $z = x + iy$, we have $z^2 - (2x)\cdot z + (x^2 + y^2) = 0$, with solutions $z' = x \pm iy$: the conjugation is automorphism, and Aut_R© = Z_2.

O) For the *octonions*, writing $o = (v, \xi)$, where v is real and ξ is a 7-vector, the vector product of the imaginary parts $\xi \wedge \xi'$ is NOT defined a priori, so the scheme cannot be applied automatically. Indeed, as shown by us somewhere else (Boya et al., 2010), octonion product maintains a quadratic form Q *plus* a 3-form ω, and indeed the Octonion Automorphism group (Cartan, 1907) is $G_2 \subset SO(7)$, an (exceptional) simple Lie group with dimension $14 = 7^2 - \{7, 3\} = 49 - 35$, consequent with G_2 keeping a 3-form in R^7. For Feynman treatment of vector products in 7 dimension (Silagadze 2002).

In a way, the work we present here is anti-historic: namely Gibbs and Heaviside introduced the vector calculus in 3 dimensions from Hamilton´s quaternions; but once this is established, we find useful look back to the quaternions proper and interpret them as a pair: scalar plus vector; after all, the concept of vector is more elementary that of quaternion... Our exposition also mimics Hamilton´s treatment of the complex numbers z as a pair of real numbers, $z = (x, y)$.

More pertinent discussions of these algebras (mainly for |H and O) can be found in the book by Conway and Smith (2003).

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