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Research Article

APPELL TYPE TRANSFORMS ASSOCIATED WITH EXPANSIONS OF DISTRIBUTIONS IN TERMS OF GENERALIZED HEAT POLYNOMIALS

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ABSTRACT
In this paper expansion of distributions in terms of the generalized heat polynomials and of their Appell transforms are studied. Two different techniques are used to prove theorems concerning expansions of distributions. A theorem which provides an orthonormal series expansion of generalized functions is also established. It is shown that this theorem gives an inversion formula for a certain generalized integral transformation.

Key Words: *Expansion Of Distributions, Generalized Heat Polynomials, Appell Transform, Orthogonal Series Expansion Of Generalized Functions And Generalized Integral Transformation.*

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1. INTRODUCTION: A procedure for expanding certain Schwartz distributions (or generalized functions) in a series of the Fourier type has developed by Zemanian [1968]. He has also shown that every distribution of polynomial growth (or tempered distribution) can be expanded in a series of Hermite functions. Some general results concerning the orthogonal series expansion of generalized functions are also available in a recent literature including reference listed by Zemanian [1968].

Kore Vaar [1959] in his paper has developed a general theory of Fourier transforms and expansions based upon the ideas different from those of the Schwartz-Sobolev approach Gelfand and Shilov [1963], and from those of Ehrenpreis [1956]. This theory did not include the notion of convergence for expansions as there exists such notions for distributions. In fact, the theory is essentially algebraic in nature. Further it is shown that distributions which are finite order derivatives of certain functions growing no faster than $e^{(ax^2)}$, $a < \frac{1}{2}$ are uniquely determined by their Hermite series. However, no topology was introduced there and hence there is no way of expressing the generalized function as a functional in terms of the Hermite coefficients.

In two papers of Waphare [2012], author studied the expansion of functions in terms of heat polynomials and their Appell transforms. This work is an extension of some results of Rosenbloom and Widder [1959] on expansions in terms of heat polynomials and associated functions. Despite these work, no attention is given to the expansion of distributions in terms of heat polynomials and their Appell transforms.

In this paper we study the expansions of distributions in terms of the generalized heat polynomials and their Appell transforms. A theorem which provides an orthogonal series expansion of generalized functions is also proved. Further it is shown that this theorem gives an inversion formula for a certain generalized integral transformation.

2. Preliminary results on the generalized heat polynomials:

For real values of x and t, the generalized heat polynomials, $P_{n,\alpha,\beta}(x,t)$ and its Appell type transform $W_{n,\alpha,\beta}(x,t)$ are defined by

$$P_{n,\alpha,\beta}(x,t) = \sum_{r=0}^{n} 2^{2r} \binom{n}{r} \frac{\Gamma(3\alpha+\beta)}{\Gamma(3\alpha+\beta+n-r)} x^{2(n-r)} t^r , \tag{2.1}$$

$$W_{n,\alpha,\beta}(x,t) = t^{-2n} G(x,t) P_{n,\alpha,\beta}(x,-t) \quad , \; t > 0, \tag{2.2}$$

where $n = 0,1,2, \dots \dots , \left(\alpha - \beta + \frac{1}{2}\right)$ is a fixed positive number and $G(x,t)$is given by

$$G(x,t) = (2t)^{-(3\alpha+\beta)} e^{-(x^2/4t)}. \tag{2.3}$$

when $\alpha - \beta = -1/2$, it follows from (2.1) that

$$P_{n,0}(x,t) = v_{2n}(x,t) \quad , \tag{2.4}$$

$$P_{n,0}(x,-1) = H_{2n}(x/2) \quad , \tag{2.5}$$

where $v_{2n}(x,t)$ is the ordinary heat polynomial of even order defined by Rosenbloom and Widder [1959, p. 222], and H_{2n} is the Hermite Polynomial of even order given in Erdelyi's book [1953].It is simple exercise to verify that for $-\infty < x,t < \infty$, $P_{n,\alpha,\beta}(x,t)$ satisfies the generalized heat equation

$$\Delta_x u(x,t) = \frac{\partial u}{\partial t} \quad , \tag{2.6}$$

where the operator

$$\Delta_x = \frac{\partial^2}{\partial x^2} + \frac{4\alpha}{x}\frac{\partial}{\partial x}$$

with some fixed positive number $\alpha - \beta + \frac{1}{2}$.

It is noted that $W_{n,\alpha,\beta}(x,t)$ is also a solution of (2.6). Further, $W_{n,\alpha,\beta}(x,t)$ satisfies the following operator formulas

$$W_{n,\alpha,\beta}(x,t) = 2^{2n} \Delta_x^n G(x,t), \tag{2.7}$$

with

$$\Delta_x^k W_{n,\alpha,\beta}(x,t) = 2^{-2k} W_{n+k,\alpha,\beta}(x,t). \tag{2.8}$$

With the help of the following results reported by Erdelyi [1956]

$$P_{n,\alpha,\beta}(x,-t) = (-1)^n 2^{2n} n! \, t^n L_n^{(\alpha-\beta)}\left(\frac{x^2}{4t}\right), \; t > 0 \tag{2.9}$$

$$|L_n^a(x)| \le \Delta_{n,a} e^{x/2} , \tag{2.10}$$

where $L_n^a\,(x)$ is the Laguerre polynomial of degree n and order $a > -1$ and

$$\Delta_{n,a} = \begin{cases} \frac{1}{n!}(a+1)_n & , \quad a \geq 0 \\ 2 - \frac{1}{n!}\,(a+1)_n & , -1 < a < 0 \end{cases} \tag{2.11}$$

with $(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)}$; it can easily be shown that for $t > 0$

$$\left|P_{n,\alpha,\beta}(x,-t)\right| \leq n!\; 2^{2n}\,\Delta_{n,\alpha-\beta}\,t^n\,e^{x^2/8t}\,, \tag{2.12}$$

$$\left|W_{n,\alpha,\beta}(x,t)\right| \leq 2^{2n-(3\alpha+\beta)} n!\,\Delta_{n,\alpha-\beta}\,t^{-(n+3\alpha+\beta)}\,e^{-x^2/8t}. \tag{2.13}$$

The differential $d\mu(x) = 2^{-(\alpha-\beta)}\,[\Gamma(3\alpha+\beta)]^{-1}\,x^{4\alpha}\,dx$ (2.14)

It is interesting to observe that $P_{n,\alpha,\beta}(x,t)$ form a biorthogonal system in $0 < x < \infty$. Indeed,

$$\int_0^\infty W_{n,\alpha,\beta}\,(x,t)\,P_{m,\alpha,\beta}\,(x,-t)\,d\mu(x) = \frac{\partial nm}{b_n}\;, t > 0, \tag{2.15}$$

where

$$b_n = \Gamma(3\alpha+\beta) / \,[2^{4n}\,n!\;\Gamma(3\alpha+\beta+n)]. \tag{2.16}$$

An important consequence of (2.15)-(2.16) is the bilinear generating function for the biorthogonal set $P_{n,\alpha,\beta}$ and $W_{n,\alpha,\beta}$ as

$$G(x,y;s+t) = \sum_{n=0}^{\infty} b_n\,W_{n,\alpha,\beta}\,(y,s)\,P_{n,\alpha,\beta}(x,t). \tag{2.17}$$

The source solution of (2.6) is given by $G(x,t)$in the form

$$G(x,y;t) = (2t)^{-(3\alpha+\beta)}\,e^{-(x^2+y^2)/4t}\,\theta\left(\frac{xy}{2t}\right) \tag{2.18}$$

$$\theta(x) = 2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)\,x^{-(\alpha-\beta)}\,I_{\alpha-\beta}(x), \tag{2.19}$$

and $I_r(x)$ is the modified Bessel function of imaginary argument of order r ; and $G(x,0;t) = G(x,t)$.

3. The test function space$U_{\sigma,\alpha,\beta}$:

We denote the open interval $(0,\infty)$ by I. A complex valued infinitely differentiable function $\phi(x)$ defined over I belongs to the space $U_{\sigma,\alpha,\beta}(I)$ if

$$\rho_k(\phi) = \sup_{0<x<\infty}\left|e^{x^2/4\sigma}\,\Delta_x^k\,\phi(x)\right| < \infty \quad , \tag{3.1}$$

for any fixed k, where k assumes the values $0,1,2,\ldots\ldots$; σ is a positive real number, $(\alpha-\beta) > -1$ and Δ_x is the differential operator considered in Section 2.

The topology in the space $U_{\sigma,\alpha,\beta}$ is induced by the collection of seminorms $\{\rho_k\}_{k=0}^{\infty}$. Since $\rho_0$is a norm, the collection of seminorms is separating as indicated by Zemanian $[1968, p.8]$.

A sequence $\{\phi_r\}_{r=1}^{\infty}$ is said to converge to ϕ in $\cup_{\sigma,\alpha,\beta}(I)$ if for each k, $\rho_k\ (\phi_r - \phi) \to 0$ as $r \to \infty$. A sequence $\{\phi_r\}_{r=1}^{\infty}$ with each $\phi_r(x)$ belonging to $\cup_{\sigma,\alpha,\beta}(I)$ is a Cauchy sequence in $\cup_{\sigma,\alpha,\beta}(I)$ if $\rho_k\ (\phi_r - \phi_s) \to 0\ r, s \to \infty$ independently of each other for every fixed k, where $k = 0,1,2, \dots\dots$. It is noted here that $\cup_{\sigma,\alpha,\beta}(I)$ is a locally convex, sequentially complete, Hausdorff topological linear space. Its dual space $U'_{\sigma,\alpha,\beta}\ (I)$ is the space of generalized functions under consideration.

From the estimate (2.10), it follows that for $x > 0$, $s > \frac{\sigma}{2}$ and $(\alpha - \beta) > -1$, $\mu(x)\,P_{n,\alpha,\beta}\,(x, -s) \in \cup_{\sigma,\alpha,\beta}(I)$. Using results (2.4), (2.5) and (2.8), it can also be seen that $W_{n,\alpha,\beta}(x,t) \in \cup_{\sigma,\alpha,\beta}(I)$ for $x > 0$ and $0 < t < \sigma$.

Now we need some properties which we list as mentioned below:

(i) $\mathcal{D}(I)$ denotes the space of infinitely differentiable functions defined over I with a compact support. The dual space $\mathcal{D}'(I)$ is the space of Schwartz distributions (see Schwartz [1950/51]) on I. It can easily be shown that $\mathcal{D}(I)$ is a subspace of the space $\cup_{\sigma,\alpha,\beta}(I)$and the topology of $\mathcal{D}(I)$ stronger than that induced on it by $\cup_{\sigma,\alpha,\beta}(I)$.

(ii) The space $\cup_{\sigma,\alpha,\beta}(I)$ is a dense subspace of $\varepsilon(I)$ which is the space of all complex-valued smooth functions on I. The topology of $U_{\sigma,\alpha,\beta}$ is stronger than the topology induced on $U_{\sigma,\alpha,\beta}$by $\varepsilon(I)$. So ε'(I) can be identified with subspace of $U'_{\sigma,\alpha,\beta}$.

(iii) The space S' of tempered distributions is a subspace of $U'_{\sigma,\alpha,\beta}$ for $(\alpha - \beta) > 0$ Pandey[1969].

(iv) For each $f \in U'_{\sigma,\alpha,\beta}$ there exists a non-negative integer r and a positive constant C such that

$$|\langle f, \phi\rangle| \le C \ \max_{0\le k\le r} \rho_k\,(\phi)\ , \tag{3.2}$$

for every $\phi \in U_{\sigma,\alpha,\beta}$, where r and C depend on f but not on ϕ $Erdelyi$ [1953, p.19].

(v) If $f(x)$ is a locally integrable for $x > 0$ such that $f(x)\ e^{-x^2/4\sigma}$ is absolutely integrable on $0 < x < \infty$, then $f(x)$ generates a regular generalized function $f \in U'_{\sigma,\alpha,\beta}$ defined by the integral

$$\langle f, \phi \rangle = \int_0^\infty f(x)\, \overline{\phi(x)}\, dx \quad , \tag{3.3}$$

where $\phi(x) \in U_{\sigma,\alpha,\beta}(I)$.

(vi) Let $f(x)$ be a locally integrable function defined for $x > 0$ such that

$$f(x) = \begin{cases} 0\,(x^\rho) & , \quad \rho + 1 > 0, \quad x \to 0+ \\ 0\left(e^{\delta x^2}\right), & 0 < \delta < 1/4\sigma \ , \ x \to \infty \end{cases}.$$

Then clearly $f \in U'_{\sigma,\alpha,\beta}$. Let $\mathcal{A}$ denote Zemanian's test function (see Zemanian [1968, p. 252]). Then

$$\phi(x) = x^{\frac{1}{2}a}\, e^{-x/2}\, L_n\,(a)_{(x)} \in \mathcal{A}.$$

But

$$\int_0^\infty f(x)\, \overline{\phi(x)}\, dx$$

does not exist. Hence, $f(x)$ satisfying (2.4) does not belong to $\mathcal{A}'$ $(Zemanian[1968, p. 258])$. Therefore, $U'_{\sigma,\alpha,\beta} \not\subset \mathcal{A}'$. Since $D(I)$ is contained in both the spaces $U_{\sigma,\alpha,\beta}$ and $\mathcal{A}$, it follows that the spaces $U'_{\sigma,\alpha,\beta}$ and $\mathcal{A}'$ overlap but $U'_{\sigma,\alpha,\beta} \not\subset \mathcal{A}'$.

4. Expansion of distributions:

The expansion of $f \in U'_{\sigma,\alpha,\beta}$ in terms of heat polynomials $P_{n,\alpha,\beta}(x, -t)$ and their Appell type transform $W_{n,\alpha,\beta}(x,t)$ is given in the following theorem:

Theorem 4.1: Let $f \in U'_{\sigma,\alpha,\beta}$ where $(\alpha - \beta) > 0$ and $\sigma > 0$. Then, for $0 < \sigma/2 < t < \sigma$,

$$\langle f(x), \phi(x) \rangle = \lim_{s \to t^-} \lim_{N \to \infty} \sum_{n=0}^{N} K_n\, a_n(t)\, \langle \mu(x)\, P_{n,\alpha,\beta}(x, -s), \ \phi(x) \rangle, \tag{4.1}$$

where

$$a_n(t) = \langle f(y) \ , \ W_{n,\alpha,\beta}(y,t) \rangle, \tag{4.2}$$

$$K_n = \Gamma(3\alpha + \beta) / \left[2^{4n}\, n!\, \Gamma(3\alpha + \beta + n)\right]. \tag{4.3}$$

We shall need the following lemma for the proof of this theorem.

Lemma 4.1: For $f \in U'_{\sigma,\alpha,\beta}$ where $(\alpha - \beta) > -\frac{1}{2}$ and $\sigma > 0$, we define

$$a_n(t) = \langle f(y), W_{n,\alpha,\beta}\,(y,t) \rangle, \ n = 0,1,2, \ldots\ldots, \tag{4.4}$$

Then for $0 < t < \sigma$,

$$|a_n(t)| \le C\,(4/t)^n\, \left[(3\alpha + \beta)_{n+r} + 2(n + r)!\right], \tag{4.5}$$

where C and r are independent of n.

Proof: From the boundedness property of generalized functions there exists a positive constant C and a non-negative integer r such that

$$\begin{aligned} |a_n| &= \left|\langle f(y), W_{n,\alpha,\beta}(y,t)\rangle\right| \\ &\leq C \max_{0\leq k\leq r} \rho_k \left(W_{n,\alpha,\beta}(y,t)\right) \\ &= C \max_{0\leq k\leq r} \sup_{0<y<\infty} \left|2^{-2k}\, e^{y^2/4\sigma}\, W_{n+k,\alpha,\beta}(y,t)\right|. \end{aligned}$$

We arrive at the desired estimate using result (2.11).

Lemma 4.2: Let $\phi(x) \in U_{\sigma,\alpha,\beta}$. Then for $0 < \sigma/2 < s < t < \sigma$,

$$e^{y^2/4\sigma} \left|\Delta_y^k \int_0^\infty \sum_{n=N}^\infty K_n\, W_{n,\alpha,\beta}(y,t)\, P_{n,\alpha,\beta}(x,-s)\, \phi(x)\, d\mu(x)\right| \to 0$$

as $N \to \infty$ uniformly for all $y > 0$.

Proof: Using the estimates (2.10) and (2.11) we can write

$$\begin{aligned} |J| &= e^{y^2/4\sigma} \left|\Delta_y^k \int_0^\infty \sum_{n=N}^\infty K_n\, W_{n,\alpha,\beta}(y,t) P_{n,\alpha,\beta}(x,-s)\, \phi(x)\, d\mu(x)\right| \\ &\leq e^{y^2/4\sigma} \int_0^\infty \sum_{n=N}^\infty K_n \frac{2^{-(\alpha-\beta)-2k}}{\Gamma(3\alpha+\beta)} \left|W_{n+k,\alpha,\beta}(y,t)\right| \left|P_{n,\alpha,\beta}(x,-s)\right| |\phi(x)| d\mu(x) \\ &\leq e^{-y^2/4\sigma\left(\frac{1}{t}-\frac{1}{\sigma}\right)} \sum_{n=N}^\infty \frac{\Gamma(3\alpha+\beta)}{\Gamma(3\alpha+\beta+n)} (n+k)!\, \Delta_{n+k,\ \alpha-\beta}\, \Delta_{n,\ \alpha-\beta}\, 2^{2k-(3\alpha+\beta)} \\ &\times t^{-2k+(3\alpha+\beta)} (s/t)^n \int_0^\infty e^{x^2/8s}\, |\phi(x)|\, d\mu(x). \end{aligned}$$

Since $\phi(x) \in U_{\sigma,\alpha,\beta}$, there exists a positive constant C such that

$$\begin{aligned} |J| &\leq C \sum_{n=N}^\infty \frac{\Gamma(4\alpha+2\beta+k+n)}{(n+1)} (s/t)^n \\ &\leq C_1 \sum_{n=N}^\infty n^{\alpha-\beta+k} (s/t)^n, \end{aligned}$$

where C_1 is another positive constant. Clearly the last term tends to zero as $N \to \infty$ for $s < t$.

Lemma 4.3: For $0 < \sigma/2 < s < t < \sigma$, let $G(x, y; t-s)$ be the function defined by (2.14) and let $\phi(x) \in U_{\sigma,\alpha,\beta}$. Then there exist $\epsilon > 0$ and a large positive constant $A_1(n)$ such that for $y > A_1(n)$,

$$e^{y^2/4\sigma}\left|\int_0^\infty \Delta_y^k\ \{G(x,y,t-s)\}\,\phi(x)\,d\mu(x)\right| < \frac{\epsilon}{2}\ ,k\ =\ 0,1,2\ldots\ldots$$

Proof: Proceeding as in the proof of Lemma 4.2, we write

$$e^{y^2/4\sigma}\left|\int_0^\infty \Delta_y^k\{G(x,y,t-s)\}\phi(x)d\mu(x)\right|$$

$$\leq\ C_2\,e^{-y^2/4\left(\frac{1}{t}-\frac{1}{\sigma}\right)}\sum_{n=0}^{\infty}\frac{\Gamma(3\alpha+\beta+k+n)}{\Gamma(n+1)}\left(\frac{s}{t}\right)^n$$

$$=\ C_3\,e^{-y^2/4\left(\frac{1}{t}-\frac{1}{\sigma}\right)}\left(1-\frac{s}{t}\right)^{-(3\alpha+\beta)-k}$$

for appropriate constants C_2 and C_3.

Thus for $t<\sigma$, the last expression tends to zero as $y\to\infty$.

Lemma 4.4: Let $A_{1,}\ \in,\ G(x,y,t-s)$ and $\phi(x)$ be the same as in Lemma 4.3 and let $(\alpha-\beta)>0$. Then for $t-s<\delta\ (n,\epsilon)$,

$$e^{y^2/4\sigma}\left|\int_0^\infty \Delta_y^k\ \{G(x,y,t-s)\}\,\phi(x)d\mu(x)-\ \phi_k(y)\right|<\epsilon\ ,$$

where $\phi_k(y)\ =\ \Delta_y^k\,\phi(y)$, uniformly for all $y\ \in\ (0,\ A_1)$.

Proof: If $\phi(x)\ \in\ U_{\sigma,\alpha,\beta}$, then proceeding as in $[5,p.846]$, it follows that for $(\alpha-\beta)>0$,

$$\phi^{(k)}\ (x)\ =\ 0\ (e^{-cx}),\qquad c>0,\qquad x\to\infty$$

$\phi^{(k)}\ (0+)$ exist finitely,

for each $k=0,1,2,\ldots\ldots$ Using these orders of $\phi^{(k)}(x)$ and the identity $\Delta_x G(x,y,t)=\Delta_y G(x,y,t)$ and integrating by parts, we obtain

$$\int_0^\infty \Delta_y\ \{G(x,y,t-s)\}\ \phi(x)d\mu(x)\ =\ \int_0^\infty G(x,y,t-s)\ \Delta_x\ \ \phi(x)\ d\mu(x).$$

Consequently

$$\int_0^\infty \Delta_y^k\ \{G(x,y,t-s)\}\,d\mu(x)-\ \phi_k(y)\ =\ \int_0^\infty G(x,y,t-s)\ [\phi_k(x)-\ \phi_k(y)]\ d\mu(x),$$

where we have made use of the fact that

$$\int_0^\infty G(x, y, t-s)\ d\mu(x) = 1.$$

By using the standard technique Pandey [1969] and remembering that $\phi(x)$ is not an element of $\mathcal{D}(I)$ although it does belong to $U_{\sigma,\alpha,\beta}$, we show that

$$e^{y^2/4\sigma}\left|\int_0^\infty G(x, y, t-s)\{\phi_k(x) - \phi_k(y)\}\, d\mu(x)\right| < \epsilon$$

uniformly for all y satisfying $0 < y \leq A_1$.

We are now prepared to prove the main expansion Theorem 4.1.

Proof of Theorem 4.1: Let t be a fixed number such that $0 < \frac{\sigma}{2} < s < t < \sigma$. Then using the estimates (2.10) and (4.5), we obtain

$$\sum_{n=0}^{\infty} k_n\, |a_n| \left|\langle \mu(x)\, P_{n,\alpha,\beta}\,(x, -s),\ \phi(x)\rangle\right|$$

$$\leq C\,\Gamma(3\alpha+\beta) \sum_{n=0}^{\infty} \frac{(3\alpha+\beta)_{n+r} + 2(n+r)!}{\Gamma(3\alpha+\beta+r)}\, \Delta_{n,\alpha-\beta} \left(\frac{s}{t}\right)^n$$

$$= C \sum_{n=0}^{\infty} \frac{\Gamma(3\alpha+\beta+r+n)}{n!} \left(\frac{s}{t}\right)^n + 2c \sum_{n=0}^{\infty} \frac{(n+r)!}{n!} \left(\frac{s}{t}\right)^n, \quad (\alpha-\beta) > 0$$

$$\leq C_1$$

where C_1 is a certain positive constant. This proves the existence of the limit $N \to \infty$, when $s < t$.

Using the linearity of $U_{\sigma,\alpha,\beta}$ and $U'_{\sigma,\alpha,\beta}$ we can write

$$\sum_{n=0}^{\infty} a_n\, k_n\, \langle \mu(x)\, P_{n,\alpha,\beta}\,(x, -s),\ \phi(x)\rangle$$

$$= \sum_{n=0}^{N} \langle f(y),\ W_{n,\alpha,\beta}\,(y,t)\rangle\, \langle \mu(x)\, k_n\, P_{n,\alpha,\beta}\,(x,-s), \phi(x)\rangle$$

$$= \sum_{n=0}^{N} \langle f(y), W_{n,\alpha,\beta,}(y,t)\rangle\, \langle \mu(x)\, k_n\, P_{n,\alpha,\beta}\,(x,-s), \phi(x)\rangle$$

$$= \langle f(y), \sum_{n=0}^{N} \langle \mu(x)\, k_n\, W_{n,\alpha,\beta}\,(y,t)\, P_{n,\alpha,\beta}\,(x,-s),\ \phi(x)\rangle\rangle$$

$$= \langle f(y), \langle \mu(x)\, G_n\,(x,y,t,s), \qquad \phi(x)\rangle\rangle$$

where

$$G_N(x,y,t,s) = \sum_{n=0}^{N} k_n\, W_{n,\alpha,\beta}\,(y,t)\, P_{n,\alpha,\beta}\,(x,-s).$$

The corresponding infinite series equals $G(x,y,t-s)$. We observe that $\mu(x)\, G(x,y,\ t-s) \in U'_{\sigma,\alpha,\beta}$ and $\mu(x)\, G_N\,(x,y,t,s) \in U'_{\sigma,\alpha,\beta}$ for $s > \frac{\sigma}{2}$. Furthermore, as a function of ,
$\langle \mu(x)\, G(x,y,t-s), \phi(x)\rangle$ is an element of $U_{\sigma,\alpha,\beta}$.

From Lemma 4.2, we know that

$$\langle \mu(x)\, G_N\,(x,y,t,s)\,, \phi(x)\rangle \to \langle \mu(x)\, G(x,y,t-s), \phi(x)\rangle$$

in $U_{\sigma,\alpha,\beta}$ as $N \to \infty$.

Thus

$$\lim_{N\to\infty} \sum_{N=0}^{N} a_n\, k_n\, \langle \mu(x)\, P_{n,\alpha,\beta}\,(x,-s),\ \ \phi(x)\rangle$$
$$= \langle f(y), \langle \mu(x)\, G(x,y,t-s), \phi(x)\rangle\rangle.$$

To complete the proof we have to show that

$$\lim_{s\to t^-} \langle \mu(x)\, G(x,y,t-s), \phi(x)\rangle = \phi(y)$$

in $U_{\sigma,\alpha,\beta}$.

In other words, we need to show that for all k and ϵ, there exists $\delta(n,\epsilon)$ such that for $0 < t - s < \delta(n,\epsilon)$, such that, for $0 < t-s < \delta(n,\epsilon)$,

$$e^{y^2/4\sigma} \left| \int_0^\infty \Delta_y^k\ \{G(x,y,t-s)\}\, \phi(x)\, d\mu(x) - \phi_k(y) \right| < \epsilon.$$

Since $\phi(y) \in U_{\sigma,\alpha,\beta}$, we have for $y > A(n)$,

$$e^{y^2/4\sigma}\, |\phi_k(y)| < \frac{\epsilon}{2}.$$

From Lemma 4.3 we know that there exists $A_1 > A$ such that for $y > A_1(n)$ and $t < t_0$ ($A_1(n)$independent of t) ,

$$e^{y^2/4\sigma} \left| \int_0^\infty \Delta_y^k\ \{G(x,y,t-s)\}\, \phi(x)\, d\mu(x) \right| < \frac{\epsilon}{2}.$$

An application of Lemma 4.4 completes the proof of the theorem.

5. The test function space $V_{\sigma,\alpha,\beta}(I)$ and its dual:

Suppose $\Omega_{x,t}$ denotes the differential operator given by

$$\Omega_{x,t} \equiv 4t\ \left[\frac{d^2}{dx^2} - \frac{x^2}{16t^2} - \beta(2\alpha - 2\beta + 1)\ \frac{1}{x^2}\right], \tag{5.1}$$

where $x \in I = (0,\infty)$ and t is a fixed real number.

A complex valued smooth function $\phi(x)$belong to the space $V_{\sigma,\alpha,\beta}\ (I)$ if

$$B_k(\phi) = \left[\int_0^\infty \left|\Omega_{x,t}^k\ \phi(x)\right|^2 dx\right]^{\frac{1}{2}} < \infty, \tag{5.2}$$

where $k = 0,1,2\ldots\ldots.$; and for each n, k

$$\left(\Omega^k\ \phi,\ \ \psi_{\mathrm{n}}\right) = \left(\phi,\ \Omega^k\ \psi_n\right), \tag{5.3}$$

then $V_{\sigma,\alpha,\beta}\ (I)$ is a linear space, and $\{B_k\}_{k=0}^{\infty}$ is a multinorm on $V_{\sigma,\alpha,\beta}$. The topology over $V_{\sigma,\alpha,\beta}\ (I)$ is a subspace of $L_2(I)$ when we identify each function in $V_{\sigma,\alpha,\beta}\ (I)$ with the corresponding equivalence class in $L_2(I)$. Also convergence in $V_{\sigma,\alpha,\beta}\ (I)$ implies convergence in $L_2(I)$. The space $V_{\sigma,\alpha,\beta}\ (I)$ is complete and therefore is Frechet. The dual space of $V_{\sigma,\alpha,\beta}\ (I)$ is denoted by $V'_{\sigma,\alpha,\beta}(I)$. It can also be shown that $V'_{\sigma,\alpha,\beta}(I)$ is sequentially complete.

We note that

$$\Omega_{x,t}\ \psi_n(x,t) = -\lambda_n\ \psi_n(x,t), \tag{5.4}$$

where

$$\psi_n\ (x,t) = \left[\frac{2^{-(\alpha-\beta)}}{\Gamma(3\alpha+\beta)}\right]^{\alpha+\beta} x^{\alpha-\beta+\frac{1}{2}}\,[G(x,t)]^{\alpha+\beta}\ P_{n,\alpha,\beta}\ (x,-t) \tag{5.5}$$

with

$$\lambda_n = 4n + 6\alpha + 2\beta\ . \tag{5.6}$$

Therefore $\psi_n\ (x,t)$, as a function of x, is a member of $V_{\sigma,\alpha,\beta}\ (I)$ for fixed t.

Following are important properties of these spaces:

(i) For fixed t, $\Omega_{x,t}$ is a continuous linear mapping of $V_{\sigma,\alpha,\beta}\ (I)$ into itself. In view of the self adjoint nature of the operator $\Omega_{x,t}$ [See(5.1)] we define the generalized differential operator $\Omega_{x,t}$ on $V'_{\sigma,\alpha,\beta}$ by

$$\left(\Omega_{x,t}\ f(x),\ \phi(x)\right) \triangleq \left(f(x),\ \Omega_{x,t}\ \phi(x)\right), \tag{5.7}$$

for fixed t, where $f \in V'_{\sigma,\alpha,\beta}$ and $\phi \in V_{\sigma,\alpha,\beta}$.

(ii) $\mathcal{D}'(I)$ is a subspace of $V_{\sigma,\alpha,\beta}(I)$ and the convergence in $\mathcal{D}(I)$ implies convergence in $V_{\sigma,\alpha,\beta}(I)$. Consequently restriction of any $f \in V'_{\sigma,\alpha,\beta}(I)$ to $\mathcal{D}(I)$ implies convergence in $\mathcal{D}'(I)$.

(iii) $V_{\sigma,\alpha,\beta}(I) \subset \varepsilon(I)$ and since $\mathcal{D}(I)$ is dense in $\varepsilon(I)$, $V_{\sigma,\alpha,\beta}(I)$ is also dense in $\varepsilon(I)$. Furthermore if $\{\phi_m\}_{m=1}^{\infty}$ converges in $V_{\sigma,\alpha,\beta}(I)$ to the limit ϕ, then $\{\phi_m\}$ also converges in $\varepsilon(I)$ to the some limit ϕ. The space $\varepsilon'(I)$ is a subspace of $V'_{\sigma,\alpha,\beta}(I)$.

(iv)We imbed $L_2(I)$ and therefore, $V_{\sigma,\alpha,\beta}(I)$ since $V_{\sigma,\alpha,\beta}(I) \subset L_2(I)$ into $V'_{\sigma,\alpha,\beta}(I)$ by defining the number that $f \in L_2(I)$ assigns to any $\phi \in V_{\sigma,\alpha,\beta}(I)$ as

$$(f,\phi) \triangleq \int_0^{\infty} f(x)\,\overline{\phi(x)}\,dx. \tag{5.8}$$

Then f is linear and continuous on $V_{\sigma,\alpha,\beta}$. This imbedding of $L_2(I)$ into $V'_{\sigma,\alpha,\beta}(I)$ is one to one.

(v) If $f(x) = \Omega_{x,t}^{k}\, g(x)$, for fixed t and $g \in L_2(I)$ and some k, then $f \in V'_{\sigma,\alpha,\beta}(I)$.

Instead of working with the number $< f,\phi >$ that $f \in V'_{\sigma,\alpha,\beta}$ assigns to $\phi \in V_{\sigma,\alpha,\beta}$ it is more convenient to work with the number that f assigns to the complex conjugate of ϕ. We write

$$(f,\phi) = \langle f,\overline{\phi}\rangle. \tag{5.9}$$

This is consistent with the linear product notation of $L_2(I)$.

The multiplication by a complex number a is given by

$$(af,\phi) = \langle af,\overline{\phi}\rangle = \langle f, a\,\overline{\phi}\,\rangle = a(f,\phi). \tag{5.10}$$

The following two lemmas are useful in the sequel.

Lemma 5.1: If $\phi(x) \in V_{\sigma,\alpha,\beta}(I)$ then for $0 < t \le \sigma$,

$$\phi(x) = \sum_{n=0}^{\infty}\big(\phi(x),\psi_n(x,t)\big)\,\psi_n(x,t), \tag{5.11}$$

where the series converges in $V_{\sigma,\alpha,\beta}(I)$.

Proof: By (5.2), $\Omega_{x,t}^{k}\,\phi(x)$ is in $L_2(I)$ for each non-negative integer k. Hence by [9] we may expand $\Omega_{x,t}^{k}\,\phi(x)$ into a series of orthogonal functions $\psi_n(x,t)$. Thus for fixed t, $0 < t \le \sigma$,

$$\Omega_{x,t}^{k}[\phi(x)] = \sum_{n=0}^{\infty}\Big(\Omega_{x,t}^{k}\,\phi(x),\psi_n\,(x,t)\Big)\psi_n(x,t)$$

$$= \sum_{n=0}^{\infty}\Big(\phi(x),\Omega_{x,t}^{k}\,\psi_n(x,t)\;\psi_n(x,t)\Big)$$

$$= \sum_{n=0}^{\infty} \left(\phi(x), \ (-\lambda_n)^k \psi_n(x,t) \right)$$

$$= \sum_{n=0}^{\infty} \left(\phi(x), \psi_n(x,t) \right) (-\lambda_n)^k \ \psi_n(x,t)$$

$$= \ \textstyle\sum_{n=0}^{\infty} (\phi, \psi_n) \ \Omega_{x,t}^k \ \psi_n. \tag{5.12}$$

Consequently, for each k,

$$B_k[\phi(x) - \ \textstyle\sum_{n=0}^{\infty} (\phi, \psi_n) \ \psi_n] \to 0, as \ N \to \infty. \tag{5.13}$$

This proves the lemma.

Using (5.3), (5.12) and the fact that the inner product is continuous with respect to each of its arguments, for any two members ϕ and X of $V_{\sigma,\alpha,\beta}$, we obtain

$$\left(\Omega_{x,t} \ \phi(x) \ , \ X(x) \right) = \ \left(\phi(x), \ \Omega_{x,t} \ X(x) \right). \tag{5.14}$$

Therefore the operator $\Omega_{x,t}$ is a self adjoint operator.

Lemma 5.2: Let a_n denote complex numbers. Then

$$\sum_{n=0}^{\infty} a_n \ \psi_n(x,t)$$

for $0 < t \leq \sigma$, converges in $V_{\sigma,\alpha,\beta}(I)$ if and only if

$$\sum_{n=0}^{\infty} |\lambda_n|^{2k} \ |a_n|^2$$

converges for every non-negative integer k.

Proof: The proof is similar to that of Zemanian $[1968, p. 255]$ and hence it is omitted.

6. Orthogonal series expansion of generalized functions:

The following theorem provides an orthonormal series expansion of generalized functions belonging to $V'_{\sigma,\alpha,\beta}$ which in turn yields an inversion formula for a certain generalized integral transformation.

Theorem 6.1: Let $F(n)$ be the generalized integral transformation of $f \in V'_{\sigma,\alpha,\beta}$ defined for fixed t by

$$F(n) \ \triangleq \ \left(f(x), \ \psi_n(x,t) \right) \ \equiv T[f]. \tag{6.1}$$

Then

$$f(x) = \sum_{n=0}^{\infty} F(n)\, \psi_n\,(x,t), \tag{6.2}$$

where the series converges in $V'_{\sigma,\alpha,\beta}\,(I)$.

Proof: By Lemma 5.1, for any $\phi(x) \epsilon\, V_{\sigma,\alpha,\beta}\,(I)$. we obtain

$$\begin{aligned}(f,\phi) &= \left(f, \sum(\phi(x),\psi_n(x,t))\psi_n\,(x,t)\right)\\ &= \sum_{n=0}^{\infty}(f,\psi_n)\,\overline{(\phi,\psi_n)}\\ &= \sum_{n=0}^{\infty}(f,\psi_n)\,(\psi_n,\phi) = \sum_{n=0}^{\infty} F_n\,\left(\psi_n(x,t),\phi(x,t)\right).\end{aligned}$$

This proves that the series on the right of (6.1) converges in $V'_{\sigma,\alpha,\beta}\,(I)$.

Theorem 6.2 (Uniqueness) : Let f and g be elements of $V'_{\sigma,\alpha,\beta}$ and let their transforms $F(n)$ and $G(n)$ satisfy $F(n) = G(n)$ for all n., then $f = g$ in the sense of equality in $V'_{\sigma,\alpha,\beta}$.

Proof: We have

$$f - g = \sum(f-g,\ \psi_n)\,\psi_n = \sum[(f,\psi_n) - (g,\psi_n)]\psi_n = 0.$$

7. Characterization Theorems: The following theorem gives a characterization of the transform $F(n)$ of $f \in V'_{\sigma,\alpha,\beta}$. Its proof being similar to that of Theorem 9. 6-1, p. 261 in $Ehrenpreis$ [1956] and hence it is omitted.

Theorem 7.1: Let b_n denote complex numbers. Then for fixed , $0 < t \le \sigma$,

$$\sum_{n=0}^{\infty} b_n\, \psi_n\,(x,t) \tag{7.1}$$

converges in $V'_{\sigma,\alpha,\beta}\,(I)$ if and only if there exists a non-negative integer q such that the series

$$\sum_{\lambda_n \neq 0} |\lambda_n|^{-2q}\ |b_n|^2$$

converges. Furthermore, if f denotes the sum in $V'_{\sigma,\alpha,\beta}$ of (7.1) then $b_n = (f,\psi_n)$.

The next theorem is analogous to that of Theorem 9.6-2, p.262 Zemanian [1968] and gives a characterization of elements of $V'_{\sigma,\alpha,\beta}(I)$.

Theorem 7.2: A necessary and sufficient condition for f to be a member of $V'_{\sigma,\alpha,\beta}(I)$is that there be some non-negative integer q and $a\ g \in L_2\,(I)$ such that for fixed t,

$$f(x) = \Omega_{x,t}^{q}\, g(x) + \sum_{\lambda_n=0} c_n\, \psi_n(x,t),\ \ 0 < t \le \sigma, \tag{7.2}$$

where c_n are certain complex constants and $\sum_{\lambda_n=0}$ denotes a summation on those n for which $\lambda_n = 0$; these are finite in number.

8. An operational calculus:

Since $\Omega_{x,t}$ is a continuous linear mapping of $V'_{\sigma,\alpha,\beta}$ for every $f \in V'_{\sigma,\alpha,\beta}$ we can write

$$\Omega_{x,t}^k f(x) = \sum_{n=0}^{\infty} \big(f(x), \psi_n(x,t)\big)\, \Omega_{x,t}^k\, \psi_n(x,t)$$

$$= \sum_{n=0}^{\infty} \big(f(x), \psi_n(x,t)\big)\, (-\lambda_n)^k\, \psi_n\,(x,t)$$

Using this fact we can solve the differential equation

$$P\big(\Omega_{x,t}\big)\, u(x) = g(x), \tag{8.2}$$

where P is a polynomial and the given g and unknown u are required to be in $V'_{\sigma,\alpha,\beta.}$

Applying the transformation T defined by (6.1) we obtain

$$P(\lambda_n)\, U(n) = G(n)\, \theta, \qquad U = Tu, G = Tg.$$

If $P(\lambda_n) \neq 0$ for every n, we can divide $\mathrm{P}(\lambda_n)$ and apply T^{-1} to obtain

$$u = \sum_{n=0}^{\infty} \frac{G(n)}{P(\lambda_n)}\, \psi_n. \tag{8.3}$$

By Theorems 6.2 and 7.1, this solution exists, and is unique in $V'_{\sigma,\alpha,\beta.}$ If $P(\lambda_n) = 0$ for some λ_n, say, for $\lambda_{n_k}\,(k = 1, \ldots\ldots, m)$, then a solution exists in $V'_{\sigma,\alpha,\beta}$ if and only if $G\big(\lambda_{n_k}\big) = 0$ for $k = 1,2 \ldots\ldots, m$. In this case solution (8.3) is no longer unique and we may add to it any complementary solution

$$u_c = \sum_{k=1}^{m} a_k\, \psi_{n_k},$$

where a_k are arbitrary constants, (8.4)

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