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Research Article

FORMULA DERIVED MATHEMATICALLY FOR COMPUTATION OF PERIMETER OF ELLIPSE

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ABSTRACT
Ellipse is one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called focus) to its distance from a fixed line (called directrix) equals to constant 'e' which is less than or equal to unity. According to Keplar's law of Planetary Motion, the ellipse is very important in geometry and the field of Astronomy, since every planet is orbiting its star in an elliptical path and its star is as one of the foci. The Perimeter is one of the important parameters of an ellipse. The results of existing formulae for calculating the values of Perimeter which are empirical formulae, inaccurately and/ or are called approximations. All these formulae results have relative error. Unlike the existing formulae, the new formula has been derived mathematically and proved its performance to be the best. In this paper, step by step derivation and its development from some basic concept of ellipse has been presented. A new theorem which defines the property of perimeter-radius of ellipse also has been developed and defined. This is an advancement in the development of more accurate formula for Perimeter of ellipse.

Key Words: Ellipse, Conic Sections, Keplar's Law of Planetary Motion, Eccentricity of ellipse, Focus, Directrix, Major axis, Minor axis, Perimeter-radius, Periapsis, Comet, Asteroids and Perimeter of an ellipse.

INTRODUTION

Here are some other common places where ellipses are found (Douglas 2003):

(i) The shape of a spotlight on a planar surface is in most cases an ellipse. In some cases it may be a circle.

(ii) If you cut a cylinder at an angle, you will get elliptical sections. This can have important applications in optics (lenses and mirrors can be elliptical in shape), or in the kitchen (where one might cut vegetables or sausage along a "bias cut" in order to obtain pieces that have the same thickness, but have more surface area.

(iii) Some tanks are in fact elliptical (not circular) in cross section. This gives them a higher capacity, but with a lower center-of-gravity, so that they are more stable when being transported. And they're shorter, so that they can pass under a low bridge. It might be seen that these tanks transporting heating oil or gasoline on the highway

(iv) The ellipse is found in art and architecture as well as and one may be familiar with the Ellipse, part of a U.S President's Park South (a National Park in Washington DC, just south of the White House).

(v) Ellipses (or semi-ellipses) are sometimes used as fins or airfoils in structures that move through the air. The elliptical shape reduces drag.

(vi) In a bicycle, you might find a chain wheel (the gear that is connected to the pedal cranks) that is approximately elliptical in shape. Here the difference between the major and minor axes of the ellipse is used to account for differences in the speed and force applied, because your legs push and pull more effectively when the pedals are arranged so that one pedal is in front and one is in back, than when the pedals are in the "dead zone" (when one pedal is up and one pedal is down).

EXISTING FORMULAE AND METHODS

There are simple formulae but they are not accurate. There are accurate formulae but they are not simple. Several attempts for more than past couple of centuries was been made by many Mathematicians to develop exact formula. However, all these are *empirical* [Borowski. E.J & Borwein. J.M (1991)] formulae and were not derived from first principles any fundamental concept related to ellipse.

For the parametric equation of ellipse $x = a\cos\theta^\circ \; and \; y = b\sin\theta^\circ$, the *formula to calculate Perimeter of an ellipse* (Zwillinger Daniel 2002) is generally defined as:

$$P = a\int_0^{2\pi}\sqrt{1 - e^2\cos^2\theta}\, d\theta \qquad ------------------------[1]$$

where, 'a' is semi-major axis and 'e' is *eccentricity of the ellipse* [Eric Weisstein (2002)] = $\sqrt{[(a^2 - b^2)/a^2]}$. The solution for the above integration for Perimeter is the *complete elliptic integral of the second kind* (Borowski and Borwein, 1991). The solution for this integral function is as infinite series

$$P = 2\pi a\left\{1 - \left[\left(\frac{1}{2}\right)^2\left(\frac{e^2}{1}\right)\right] - \left[\left(\frac{1\times 3}{2\times 4}\right)^2\left(\frac{e^4}{3}\right)\right] - \cdots - \left[\left(\frac{(2n-1)!}{2n!}\right)^2\left(\frac{e^{2n}}{2n-1}\right)\right]\ldots\right\} \quad ---[2]$$

The main disadvantage of this formula is that the degree of accuracy depends upon the number of terms taken into account for calculation. Following are some of such formulae (Michon, 2000, 2011):

(i) The first empirical formula in 1742 by Colin Maclaurin

$$(1)\; P \approx 2\pi a\left[1 - \left(\frac{1}{4}\right)e^2 - \left(\frac{3}{64}\right)e^{24} - \left(\frac{5}{256}\right)e^6 - \cdots\left(\frac{1}{2n-1}\times\left[\frac{2n!}{(2^n n!)^2}\right]^2\right)e^{2n} - ..\right] \quad --[3]$$

$$\text{where, } e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$(2)\; P \approx 2\pi a\left[1 - \left(\frac{1}{4}\right)e^2 - \left(\frac{3}{64}\right)e^4 - \left(\frac{5}{256}\right)e^6 - \left(\frac{85}{8192}\right)e^8 - \left(\frac{231}{32768}\right)e^{10} - \cdots\right] \quad ---[4]$$

but it having relative error about 3.727×10^{-13}

(ii) Ramanujam & Lindner in 1914

$$P \approx \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right] \approx \pi(a+b)\left[3 - \sqrt{4-h}\right] \qquad --------[5]$$

$$\text{where, } h = \left(\frac{a-b}{a+b}\right)^2$$

(ii) Lindner

$$P \approx \pi(a+b)\left[1 + \left(\frac{h}{8}\right)\right]^2 \qquad -------------------------[6]$$

(iii) Ramanujam in 1914

$$P \approx \pi(a+b)\left[1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right] \qquad ---------------------[7]$$

(iv) Ralph G. Hudson in 1917

$$P \approx \pi(a+b)\left[\frac{64 - 3h^2}{64 - 16h}\right] \qquad -----------------------[8]$$

(v) Gauss-Kummer series in 1917

$$P \approx \pi(a+b)\left[1 + \frac{h}{4} + \frac{h^2}{64} + \frac{h^3}{256} + \cdots\right] \qquad ---------------[9]$$

(vi) Padé approximant by Jacob & Waadeland in 1985

$$P \approx \pi(a+b)\left[\frac{256-48h-21h^2}{256-112h-21h^2}\right] \quad ----------------------[10]$$

(vii) Padé approximant altered by Charles Hermite in 1882-1901

$$P \approx \pi(a+b)\left[\frac{3072-1280h-252h^2+33h^3}{3072-2048h-212h^2}\right] \quad ------------------[11]$$

(viii) Ernst S. Selmer

$$P \approx \pi(a+b)\left[\frac{16+3h}{16-3h}\right] \quad ------------------------[12]$$

(ix) Hudson's formula

$$P \approx \pi(a+b)\left[\frac{64+16h}{64-h^2}\right] \quad ------------------------[13]$$

(x) Peanos formula in 1889

$$P \approx \pi\left[\frac{3(a+b)}{2}-\sqrt{ab}\right] \approx \frac{\pi}{4}(a+b)\left[\frac{3-\sqrt{1-h}}{2}\right] \quad ---------------[14]$$

(xi) 'YNOT' formula by Roger Maertens in 1959

$$P \approx 4(a^y+b^y)^{\frac{1}{y}}\ P \approx 4a\left[1+(1-e^2)^{\frac{y}{2}}\right]^{\frac{1}{y}} \quad --------------[15]$$

where, $y = \dfrac{\ln(2)}{\ln\left(\frac{\pi}{2}\right)}$

(xii) Euler's formula in 1773

$$P \approx \pi\sqrt{2(a^2+b^2)} \quad --------------------------[16]$$

(xiii) Takakazu Seki Kowa formula

$$P \approx \pi(a+b)\sqrt{1+h\left(\frac{16}{\pi^2}-1\right)} \quad ----------------------[17]$$

(viv) Cantrell's formula in 2001

$$P \approx 4(a+b)-\frac{2(4-\pi)ab}{H_p} \approx 4(a+b)-2(4-\pi)H_{-p} \quad -------------[18]$$

where, Holder mean of the principal radii $(H_p) = \left[\dfrac{(a^p+b^p)}{2}\right]^{\frac{1}{p}}$ ---------- [19]

$$H_{-p} = \frac{ab}{H_p}$$

(xv) Kepler's formula by Johannes Kepler in 1609

$$P \approx 2\pi\sqrt{ab} \quad ----------------------------[20]$$

(xvi) Muir's formula by Thomas Muirin in 1883

$$P \approx 2\pi\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \quad --------------------------[21]$$

where, $p = \dfrac{3}{2}$

(xvii) Cayley's series in 1876

$$P \approx 4a\left\{1+\frac{x}{4}\left[\ln\left(\frac{16}{x}\right)-1\right]+\frac{3x^2}{32}\left[\ln\left(\frac{16}{x}\right)-\frac{13}{6}\right]+\frac{15x^3}{256}\left[\ln\left(\frac{16}{x}\right)-\frac{12}{5}\right]+\cdots\right\} \quad ---[22]$$

(xviii) David W. Cantrell in 2004

$$(1)\ P \approx \pi(a+b)\left[1+\frac{3h}{10+\sqrt{4-3h}}+4h^6\left(\frac{1}{\pi}-\frac{7}{22}\right)f\right] \quad ---[23]$$

where, $f = h^6$

$$(2)\ P \approx 4(a+b)-\frac{2(4-\pi)ab}{f} \quad ---[24]$$

where, $f = p(a+b)+\left[\frac{1-2p}{k+1}\right]\sqrt{(a+kb)(ka+b)}$

$k = 133$ & $p = 0.412$ approximately

(xix) Ricardo Bartolomeu in 2004

$$(1)\ P \approx \pi(a-b)\tan^{-1}\left(\frac{a-b}{a+b}\right) \approx \pi(a+b)\left[1+\frac{h}{3}+O(h)^2\right] \quad ---[25]$$

$$(2)\ P \approx \pi\sqrt{2(a^2+b^2)}\left(\frac{\sin x}{x}\right) \quad ---[26]$$

where, $x = \left[1-\left(\frac{b}{a}\right)\right]\frac{\pi}{4}$

(xx) E. H. Lockwood in 1932

$$(1)\ P \approx \frac{4b^2}{a}\tan^{-1}\left(\frac{a}{b}\right)+\frac{4a^2}{ba}\tan^{-1}\left(\frac{b}{a}\right) \approx \pi(a+b)\left[1+\left(4-\frac{12}{\pi}\right)h+\cdots\right] \quad ---[27]$$

$$(2)\ P \approx \pi(a+b)\left[1+\left(4-\frac{12}{\pi}\right)h\right] \quad ---[28]$$

(xxi) Khaled Abed in 2009

$$P \approx \pi(a+b)\left(1+\frac{h}{4}\right)^{\frac{1}{2}+kh^4}+\left(1+\frac{h^2}{16}\right)^{-\frac{1}{4}}\left(1-\frac{h}{4}\right)^{-\frac{1}{2}} \quad ---[29]$$

(xxii) Zafary's Formula in 2009

$$(1)\ P \approx 4(a+b)\left(\frac{\pi}{4}\right)^{4ab/(a+b)^2} \quad ---[30]$$

$$(2)\ P \approx 4(a+b)\left(\frac{4}{\pi}\right)^{h} \quad ---[31]$$

(xxiii) David F. Rivera's Formula in 2004

$$(1)\ P \approx 4\left[\frac{\pi ab+(a-b)^2}{a+b}\right]-\frac{89}{146}\left(\frac{a\sqrt{b}-b\sqrt{a}}{a+b}\right)^2 \quad ---[32]$$

$$(2)\ P \approx \pi(a+b)\left[1+\left(\frac{4\pi-1}{h}\right)-\frac{p}{4\pi}\left(1-h-[1-h]^{\frac{3}{2}}\right)\right] \quad ---[33]$$

$$(3)\ P \approx 2a\left[2+(\pi-2)\left(\frac{b}{a}\right)^{1.456}\right] \quad ---[34]$$

(xxiv) Lu Chee Ket Formula in 2004

$$P \approx \pi\sqrt{2(a^2+b^2)}\left(1-\frac{\delta}{2^4}-\frac{15\delta^2}{2^{10}}-\frac{105\delta^3}{2^{16}}-\cdots\right) \quad ---[35]$$

where, $\delta = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 = \frac{4h}{1 + h^2}$

Thus, the effort was being continued for more than couple of centuries.

NECESSITY & IMPORTANCE OF PRECISE FORMULA

The *earlier formula* (Rod Pierce, 2011) to calculate Perimeter of an ellipse is generally an infinite series defined as:

$$P = 2\pi a\left\{1 - \left[\left(\frac{1}{2}\right)^2 \times \left(\frac{e^2}{1}\right)\right] - \left[\left(\frac{1 \times 3}{2 \times 4}\right)^2 \times \left(\frac{e^4}{3}\right)\right] - \left[\left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6}\right)^2 \times \left(\frac{e^6}{5}\right)\right] - \cdots\right\} \quad --[36]$$

The problem with above eqn. [36] is a transcendental function and its evaluation through infinite series or fractions is computationally inefficient, especially when implemented on simple microprocessors using floating point emulation software or hard-wired in FPGA's (Field Programmable Gate Arrays). For this reasons, various authors have proposed a number of approximations to Perimeter using simple algebraic expressions or, at most, formulae containing only commonly used functions such as square root, generic powers, arctangent, etc.,

Rather strangely, the perimeter of an ellipse is very difficult to calculate. As described in the previous section, it seems that all mathematicians found it difficult to find out an exact formula for it. Now the author has also attempted but in a different manner such that in spite of attempting as empirical effort the derivation of formula started from the fundamental logic related to ellipse. It is to be examined whether the long journey of approach for a precise formula was now come to an end by the invention of this new formula.

BASIC CONCEPTUAL LOGIC & DEVELOPMENT OF FORMULA

Basic Concept from some fundamental logic for development of the formula

In this regard, the author has also made several attempts for past more than couple of years. Finally he has made an attempt to study the pattern, curve & variation of perimeter-radius (r_p) and Perimeter. The author has plotted a graph of quarter portion of ellipse (Fig.1) based on the following operation and curve by connecting all the extreme points of all perimeter-radius corresponding to the various values of $a \,\&\, b \text{ such that } (0 \leq b \leq a)$ and 'a' is as constant. The value of 'a' is limited to 15 units so that the graph pattern can be seen very clearly. The values of Perimeter of each value of 'b' are worked out by the *following expansion* (Rod Pierce, 2011)

$$P = 2\pi a\left\{1 - \left[\left(\frac{1}{2}\right)^2\left(\frac{e^2}{1}\right)\right] - \left[\left(\frac{1 \times 3}{2 \times 4}\right)^2\left(\frac{e^4}{3}\right)\right] - \cdots - \left[\left(\frac{1 \times 3 \times \ldots 21}{2 \times 4 \times \ldots 22}\right)^2\left(\frac{e^{22}}{21}\right)\right]\right\} \quad ---[37]$$

The values of Perimeter-radius 'r_p' were worked out from its Perimeter. The values are presented in the table-1. The curve by connecting all the extreme points of all perimeter-radius 'r_p' corresponding to the various values of 'b' also plotted in the same graph. The same was studied thoroughly with respect to the variation of 'r_p' .

After plotting the graph, the author has observed that the Perimeter -radius (r_p) is always such that: $r_p = \frac{2a}{\pi}$ when $b = 0$ and $r = \frac{a}{\sqrt{2}}$ when $b = a$. It is an important observation which paved way for the idea to develop the new formula. This can be described such that in any ellipse it is true that: If $b \to 0$, Perimeter $(P) \to 4a$ and if $b \to a$, Perimeter $(P) \to 2\pi a$. Let, Perimeter (P) = $2 \times \pi \times r_p$

$$\therefore r_p = \frac{P}{2\pi} \quad -------------------------[38]$$

Where, the value of P is obtained from eqn. [37]

$$\text{Therefore: (i) if } b \to 0, r_p \to \frac{4\pi}{2a} \therefore \text{ if } b = 0, r_p = \frac{2\pi}{a} \qquad ------------[39]$$

$$\text{Similarly: (ii) if } b \to a, r_p \to \frac{2\pi a}{2\pi} \therefore \text{ if } b = a, r_p = a \qquad ------------[40]$$

$$\text{Let, } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Therefore: (iii) if } b \to 0, \theta \to 0 \therefore \text{ if } b = 0, \theta = 0 \qquad ------------[41]$$

$$\text{Therefore: (iv) if } b \to a, \theta \to 0 \therefore \text{ if } b = a, \theta = \frac{\pi}{4} \qquad ------------[42]$$

$$\text{From [2] \& [4] (i) if } b = 0, \quad x = r_p \times \cos\theta = \left(\frac{2\pi}{a}\right) \times \cos(0) = \frac{2\pi}{a} \qquad -----[43]$$

$$\text{From [3] \& [5] (i) if } b = a, \quad x = r_p \times \cos\theta = (a) \times \cos\left(\frac{\pi}{4}\right) = \frac{a}{\sqrt{2}} \qquad ------[44]$$

Derivation of formula

As per discussion above, for any value of 'b' and keeping the value of 'a' is constant, it is well confirmed that

$$\frac{2a}{\pi} < x < \frac{a}{\sqrt{2}}$$

Moreover, it is linear variation, since the power of ‘a’ is unity. Fig.2 is a graph defining the above value of 'x'. In this fig., point 'A' is the origin, AH is the lowest value of 'x', BD is the highest value of 'x', horizontal axis represents the 'b' value and vertical axis represents the 'x' value.

$b = 0$ at point A and $b = a$ at point B.

$$AB = a \qquad ------------------------------[45]$$

$$AH = \frac{2a}{\pi} \qquad ------------------------------[46]$$

$$BD = \frac{a}{\sqrt{2}} \qquad ------------------------------[47]$$

$$AE = b \qquad ------------------------------[48]$$

In trapezium AHBD, $\quad CD = BD - BC$

$\therefore CD = BD - AH \quad (\because BC = AH)$

Substituting [47] and [46] in above relation, we get

$$CD = \frac{a}{\sqrt{2}} - \frac{2a}{\pi}$$

$$\therefore CD = \frac{a\pi - 2a}{\pi\sqrt{2}}$$

$$\therefore CD = \frac{a(\pi - 2)}{\pi\sqrt{2}} \qquad --------------------------[49]$$

By using the graph, the 'x' value corresponding to any value of 'b' can be calculated by *linear-interpolation*. In right angled triangle HCD. Let, $HF = b, EG = x, EF = AH = x'$ and $FG = x''$

$$\frac{CD}{HC} = \frac{FG}{HF}$$

$$\therefore FG = \frac{CD}{HC} \times HF$$

$$\therefore FG = \frac{CD}{AB} \times HF$$

$$\therefore x'' = \frac{CD}{AB} \times AE$$

Substituting [49], [45] and [46] in above relation

$$x'' = \left(\frac{\left[\frac{a(\pi - 2)}{\pi\sqrt{2}} \right]}{a} \right) \times b \qquad ----------------------------- [50]$$

In the fig.2, $x = x' + x''$

Substituting [50] and [46] in above relation

$$x = \frac{2a}{\pi} + \left[\frac{\left(\frac{a(\pi - 2)}{\pi\sqrt{2}} \right)}{a} \times b \right]$$

Simplifying the above eqn. we get $x = \dfrac{2\sqrt{2}(a - b) + \pi b}{\pi\sqrt{2}}$ ------------ [51]

Referring fig.3, 'O' is the origin, $OQ = x$, $QP = y$ and $OP = r_p$. The *equation of ellipse* [Zwillinger Daniel (2002)] with respect to *Cartesian co-ordinates* [Eric Weisstein (2002)] (x, y) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The above eqn. defines an ellipse can be traced as a closed curve that lies entirely between the lines $(x = +a)$ & $(x = -a)$ and the lines $(y = +b)$ & $(y = -b)$ with semi-major axis 'a' and semi-major axis 'b' corresponding to $x - axis$ and $y - axis$ respectively, a & b are assumed to be non-negative real numbers. For the purpose of the analysis, it may be assumed that $a > b$.

$$\therefore y^2 = \left(1 - \frac{x^2}{a^2}\right) b^2$$

$$\therefore y^2 = \left(\frac{a^2 - x^2}{a^2}\right) b^2 \qquad --------------------------- [52]$$

We know already that the extreme point of perimeter-radius is one of the points of the ellipse. Therefore, substituting the eqn. [51] in eqn.[52], we get

$$\therefore y^2 = \left(\frac{a^2 - \left[\frac{2\sqrt{2}(a - b) + \pi b}{\pi\sqrt{2}} \right]^2}{a^2} \right) b^2$$

On rewritting, we get $y^2 = \left(a^2 - \left[\dfrac{2\sqrt{2}(a - b) + \pi b}{\pi\sqrt{2}} \right]^2 \right) \dfrac{b^2}{a^2}$ ---------- [53]

In right – angled triangle OPQ (fig. 3), $OP^2 = OP^2 + QP^2$

$\therefore {r_p}^2 = x^2 + y^2$. Substituting [51] and [53] in above eqn, we get

$${r_p}^2 = \left[\frac{2\sqrt{2}(a - b) + \pi b}{\pi\sqrt{2}} \right]^2 + \left(a^2 - \left[\frac{2\sqrt{2}(a - b) + \pi b}{\pi\sqrt{2}} \right]^2 \right) \frac{b^2}{a^2}$$

$$r_p{}^2 = \left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2 + b^2 - \frac{b^2}{a^2}\left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2$$

$$r_p{}^2 = \left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2\left(1-\frac{b^2}{a^2}\right) + b^2$$

$$r_p{}^2 = \left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2\left(\frac{a^2-b^2}{a^2}\right) + b^2$$

$$r_p = \sqrt{\left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2\left(\frac{a^2-b^2}{a^2}\right) + b^2}$$

$$r_p = \sqrt{b^2 + \left\{e\left[\frac{2(a-b)}{\pi} + \frac{b}{\sqrt{2}}\right]\right\}^2} \quad \text{-----------------------} [54]$$

The eqn. [53] is the Maran's formula arrived at to calculate the perimeter-radius of any ellipse.
As discussed in earlier section, the Perimeter of an ellipse = $2\pi r_p$
Substituting, eqn. [54] in above, we get

$$\text{Perimeter of ellipse (P)} = 2\pi\sqrt{b^2 + \left\{\left(\frac{a^2-b^2}{a^2}\right)\times\left[\frac{2(a-b)}{\pi} + \frac{b}{\sqrt{2}}\right]^2\right\}} \quad \text{(or)}$$

$$\text{Perimeter of ellipse (P)} = 2\pi\sqrt{b^2 + \left\{e\left[\frac{2(a-b)}{\pi} + \frac{b}{\sqrt{2}}\right]\right\}^2} \quad \text{------------} [55]$$

where, e is the eccentricity of the ellipse. The eqn. [55] is the formula to calculate the Perimeter of any ellipse.

$$\text{In fig. 3, } \tan\theta = \frac{PQ}{OQ}$$

$$\therefore \tan\theta = \frac{y}{x}$$

Substituting [51]and [53]in above eqn, we get

$$\tan\theta = \frac{\frac{b}{a}\sqrt{a^2 - \left[\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right]^2}}{\left(\frac{2\sqrt{2}(a-b)+\pi b}{\pi\sqrt{2}}\right)}$$

$$\text{Simplifying, we get } \tan\theta = \frac{b}{a}\sqrt{\frac{2a^2\pi^2 - \left[2\sqrt{2}(a-b)+\pi b\right]^2}{\left[2\sqrt{2}(a-b)+\pi b\right]^2}}$$

$$\therefore \theta_p = \tan^{-1}\left(\frac{b}{a}\sqrt{\frac{2a^2\pi^2 - \left[2\sqrt{2}(a-b)+\pi b\right]^2}{\left[2\sqrt{2}(a-b)+\pi b\right]^2}}\right) \quad \text{-------------} [56]$$

Where, θ_p is angle of perimeter- radius with periapsis (Fig.3)The eqn. [56] is the formula to calculate the slope of the perimeter-radius of any ellipse. Another *formula for radius of ellipse* [Zwillinger Daniel (2002)] in polar coordinates form at an angle $\theta°$ with *periapsis* [Eric Weisstein (2002)] is

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \theta° + b^2 \cos^2 \theta°}}$$

By substituting the values obtained from eqn. [56] in above eqn., the perimeter-radius can also be calculated from the following formula.

$$r_p = \frac{ab}{\sqrt{a^2 \sin^2 \theta_p + b^2 \cos^2 \theta_p}} \quad \text{--------------------} [57]$$

NEW (MARAN'S) THEOREM ON PERIMETER-RADIUS OF ELLIPSE

If the Perimeter of an ellipse is defined in terms of its perimeter – radius (r_p),

then (i) the value of $r_p = \frac{2a}{\pi}$ *when* $b = 0$, *(ii) the value of* $r_p = a$ *when* $b = a$.

(iii) For other intermediate values of 'b', the value of (r_p) *is equal to the value calculated by linear – interpolation of (i) and (ii).*

Where, 'a' = semi – major axis and 'b' = semi – minor axis

RESULTS AND DISCUSSION

Table-2 is comparison of the results obtained by existing exact formula (expansion i.e. eqn. 36) and new formula (eqn. 55) for a = 15 and various values of b = 0 to 15.
In this table, Column 1 is the serial numbers
Column 2 is the values of semi major axis 'a'
Column 3 is the values of semi minor axis 'b'
Column 4 is the values of Perimeter 'P' calculated by using the existing formula (eqn. 37)
Column 5 is the values of perimeter-radius 'r_p' corresponding to col.4 (eqn. 38)
Column 6 is the values of 'θ_p' corresponding to col.4 but was obtained from the graph (fig.1)
Column 7 is the values of 'r_p' calculated by using the new formula (eqn. 54)
Column 8 is the values of 'P' calculated by using the New formula (eqn. 55) and
Column 9 is the values of Perimeter 'θ_p' calculated by using the new formula (eqn. 56).

From the table, it can be clearly seen that the result of new formula is very close to result of existing series. The result of existing expansion for 'P' may vary due to not taking of more terms in the series. For example:

(i) for $a = 15$ & $b = 0$, the actual result of $P = 4a$ it should be 60 instead of 61.312 and

(ii) r_p should be $= \frac{P}{2\pi} = 9.549$ instead of 9.758

(iii) Similarly, for $a = 15, b = 0$, the actual result of $P = 2\pi a$ it should be 94.248 instead of 94.238, since it is a circle and

(iv) r_p should be equal to $a = 15$ instead of 14.999. These are exactly correct in result obtained by the new formula only.

Table-3 is the comparison of results calculated by various formulae.

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Table 1: The values of Perimeter worked out by eqn.37

S.No	Dimensions of ellipse		Calculated by eqn. [37]		
	A	b	P	r_p	θ_p
(1)	(2)	(3)	(4)	(5)	(6)
1	15.0	0.000	61.312	9.758	0.000
2	15.0	0.050	61.312	9.758	0.223
3	15.0	0.125	61.316	9.759	0.557
4	15.0	0.250	61.327	9.761	1.115
5	15.0	0.500	61.374	9.768	2.227
6	15.0	1.000	61.562	9.798	4.443
7	15.0	1.500	61.871	9.847	6.633
8	15.0	2.000	62.298	9.915	8.786
9	15.0	2.500	62.836	10.001	10.891
10	15.0	3.000	63.479	10.103	12.945
11	15.0	3.500	64.218	10.221	14.938
12	15.0	4.000	65.047	10.353	16.864
13	15.0	5.000	66.936	10.653	20.511
14	15.0	6.000	69.081	10.995	23.893
15	15.0	7.000	71.429	11.369	27.016
16	15.0	8.000	73.937	11.768	29.896
17	15.0	9.000	76.574	12.187	32.564
18	15.0	10.000	79.319	12.624	34.809
19	15.0	11.000	82.157	13.076	37.329
20	15.0	12.000	85.076	13.541	39.470
21	15.0	13.000	88.067	14.017	41.461
22	15.0	14.000	91.124	14.503	43.351
23	15.0	14.500	92.674	14.750	44.161
24	15.0	15.000	94.238	14.999	45.000

Table 2: Comparison of Perimeter worked existing and new formulae

S.No	Dimensions of ellipse		Calculated by existing formula			Calculated by Maran's formula		
			37	38	Fig.1	54	55	56
	a	B	P	r_p	θ_p	P	r_p	θ_p
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	15.0	0.000	61.312	9.758	0.000	60.000	9.549	0.000
2	15.0	0.050	61.312	9.758	0.223	60.023	9.553	0.231
3	15.0	0.125	61.316	9.759	0.557	60.058	9.559	0.577
4	15.0	0.250	61.327	9.761	1.115	60.123	9.569	1.153
5	15.0	0.500	61.374	9.768	2.227	60.270	9.592	2.298
6	15.0	1.000	61.562	9.798	4.443	60.635	9.650	4.560
7	15.0	1.500	61.871	9.847	6.633	61.092	9.723	6.780
8	15.0	2.000	62.298	9.915	8.786	61.637	9.810	8.953
9	15.0	2.500	62.836	10.001	10.891	62.266	9.910	11.073

10	15.0	3.000	63.479	10.103	12.945	62.977	10.023	13.137
11	15.0	3.500	64.218	10.221	14.938	63.763	10.148	15.140
12	15.0	4.000	65.047	10.353	16.864	64.622	10.285	17.082
13	15.0	5.000	66.936	10.653	20.511	66.540	10.590	20.772
14	15.0	6.000	69.081	10.995	23.893	68.695	10.933	24.202
15	15.0	7.000	71.429	11.369	27.016	71.056	11.309	27.373
16	15.0	8.000	73.937	11.768	29.896	73.592	11.713	30.294
17	15.0	9.000	76.574	12.187	32.564	76.276	12.140	32.979
18	15.0	10.000	79.319	12.624	34.809	79.082	12.586	35.442
19	15.0	11.000	82.157	13.076	37.329	81.988	13.049	37.699
20	15.0	12.000	85.076	13.541	39.470	84.974	13.524	39.768
21	15.0	13.000	88.067	14.017	41.461	88.023	14.009	41.665
22	15.0	14.000	91.124	14.503	43.351	91.119	14.502	43.404
23	15.0	14.500	92.674	14.750	44.161	92.680	14.750	44.219
24	15.0	15.000	94.238	14.999	45.000	94.248	15.000	45.000

Table 3: Overall Comparison of Perimeter worked out by various formulae

S.No	a	b	Values of perimeter obtained by the formula					
			Eqn. 2	Eqn. 5	Eqn. 6	Eqn. 7	Eqn. 8	Eqn. 10
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
1	15.0	0.000	60.4874	59.7507	59.6412	59.9759	59.8866	59.9467
2	15.0	0.050	60.4882	59.7693	59.6640	59.9823	59.8995	59.9559
3	15.0	0.125	60.4923	59.8001	59.7008	59.9963	59.9224	59.9736
4	15.0	0.250	60.5072	59.8593	59.7692	60.0308	59.9692	60.0130
5	15.0	0.500	60.5665	60.0050	59.9309	60.1374	60.0941	60.1262
6	15.0	1.000	60.8008	60.3972	60.3470	60.4781	60.4560	60.4734
7	15.0	1.500	61.1821	60.9083	60.8743	60.9589	60.9473	60.9569
8	15.0	2.000	61.6982	61.5232	61.5002	61.5553	61.5491	61.5545
9	15.0	2.500	62.3343	62.2295	62.2140	62.2501	62.2467	62.2497
10	15.0	3.000	63.0750	63.0168	63.0064	63.0301	63.0282	63.0299
11	15.0	3.500	63.9057	63.8766	63.8696	63.8851	63.8841	63.8850
12	15.0	4.000	64.8135	64.8014	64.7968	64.8069	64.8064	64.8069
13	15.0	5.000	66.8199	66.8222	66.8202	66.8245	66.8243	66.8245
14	15.0	6.000	69.0324	69.0384	69.0376	69.0393	69.0393	69.0393
15	15.0	7.000	71.4118	71.4190	71.4187	71.4194	71.4194	71.4194
16	15.0	8.000	73.9319	73.9397	73.9396	73.9398	73.9398	73.9398
17	15.0	9.000	76.5729	76.5810	76.5809	76.5810	76.5810	76.5810
18	15.0	10.000	79.3188	79.3272	79.3272	79.3272	79.3272	79.3272
19	15.0	11.000	82.1567	82.1654	82.1654	82.1655	82.1655	82.1655
20	15.0	12.000	85.0760	85.0850	85.0850	85.0850	85.0850	85.0850
21	15.0	13.000	88.0675	88.0768	88.0768	88.0768	88.0768	88.0768
22	15.0	14.000	91.1236	91.1333	91.1333	91.1333	91.1333	91.1333
23	15.0	14.500	92.6738	92.6836	92.6836	92.6836	92.6836	92.6836
24	15.0	15.000	94.2378	94.2478	94.2478	94.2478	94.2478	94.2478

Table 3: Overall Comparison of Perimeter worked out by various formulae (contd...)

S.No	a	b	Values of perimeter obtained by the formula						
			Eqn. 12	Eqn. 13	Eqn. 14	Eqn. 16	Eqn. 17	Eqn. 20	Eqn. 21
[1]	[2]	[3]	[10]	[11]	[12]	[13]	[14]	[15]	[16]
1	15.0	0.000	59.6903	59.8399	70.6858	66.6432	60.0000	0.0000	59.3724
2	15.0	0.050	59.7113	59.8553	68.2008	66.6436	60.0471	5.4414	59.3800
3	15.0	0.125	59.7455	59.8816	66.9731	66.6456	60.1188	8.6036	59.4025
4	15.0	0.250	59.8099	59.9336	65.7803	66.6525	60.2415	12.1673	59.4575
5	15.0	0.500	59.9646	60.0670	64.4384	66.6803	60.4987	17.2072	59.6130
6	15.0	1.000	60.3701	60.4402	63.2309	66.7912	61.0588	24.3347	60.0518
7	15.0	1.500	60.8902	60.9380	62.8525	66.9756	61.6787	29.8038	60.6176
8	15.0	2.000	61.5110	61.5437	62.9034	67.2330	62.3567	34.4144	61.2842
9	15.0	2.500	62.2213	62.2435	63.2286	67.5625	63.0907	38.4765	62.0359
10	15.0	3.000	63.0114	63.0264	63.7486	67.9630	63.8790	42.1489	62.8619
11	15.0	3.500	63.8730	63.8830	64.4162	68.4334	64.7195	45.5260	63.7538
12	15.0	4.000	64.7990	64.8058	65.2007	68.9721	65.6103	48.6693	64.7050
13	15.0	5.000	66.8212	66.8241	67.0408	70.2481	67.5343	54.4140	66.7643
14	15.0	6.000	69.0380	69.0392	69.1564	71.7770	69.6352	59.6075	69.0051
15	15.0	7.000	71.4189	71.4193	71.4808	73.5428	71.8974	64.3835	71.4007
16	15.0	8.000	73.9396	73.9398	73.9705	75.5290	74.3062	68.8288	73.9302
17	15.0	9.000	76.5809	76.5810	76.5953	77.7187	76.8478	73.0040	76.5764
18	15.0	10.000	79.3272	79.3272	79.3332	80.0952	79.5095	76.9530	79.3252
19	15.0	11.000	82.1654	82.1655	82.1676	82.6424	82.2796	80.7090	82.1647
20	15.0	12.000	85.0850	85.0850	85.0856	85.3450	85.1476	84.2978	85.0848
21	15.0	13.000	88.0768	88.0768	88.0769	88.1887	88.1039	87.7399	88.0768
22	15.0	14.000	91.1333	91.1333	91.1333	91.1603	91.1398	91.0520	91.1333
23	15.0	14.500	92.6836	92.6836	92.6836	92.6903	92.6853	92.6637	92.6836
24	15.0	15.000	94.2478	94.2478	94.2478	94.2478	94.2478	94.2478	94.2478

Table 3: Overall Comparison of Perimeter worked out by various formulae (contd...)

S.No	a	b	Values of perimeter obtained by the formula						
			Eqn. 23	Eqn. 25	Eqn. 26	Eqn. 28	Eqn. 31	Eqn. 34	Eqn. 55
[1]	[2]	[3]	[17]	[18]	[19]	[20]	[21]	[22]	[23]
1	15.0	0.000	56.6452	60.0000	60.0000	55.6194	60.0000	60.0000	60.0000
2	15.0	0.050	56.6698	60.0549	60.0432	55.6919	60.0077	60.0085	60.0226
3	15.0	0.125	56.7137	60.1381	60.1090	55.8021	60.0228	60.0322	60.0584
4	15.0	0.250	56.8031	60.2790	60.2214	55.9894	60.0569	60.0882	60.1229
5	15.0	0.500	57.0300	60.5697	60.4565	56.3772	60.1577	60.2421	60.2699
6	15.0	1.000	57.6219	61.1860	60.9682	57.2035	60.4772	60.6641	60.6348
7	15.0	1.500	58.3468	61.8487	61.5359	58.0921	60.9345	61.1985	61.0916
8	15.0	2.000	59.1745	62.5579	62.1602	59.0375	61.5103	61.8220	61.6367

9	15.0	2.500	60.0874	63.3131	62.8412	60.0348	62.1890	62.5214	62.2664
10	15.0	3.000	61.0726	64.1139	63.5787	61.0796	62.9577	63.2880	62.9767
11	15.0	3.500	62.1201	64.9596	64.3724	62.1682	63.8058	64.1153	63.7634
12	15.0	4.000	63.2221	65.8495	65.2215	63.2971	64.7243	64.9985	64.6223
13	15.0	5.000	65.5637	67.7582	67.0820	65.6637	66.7433	66.9174	66.5396
14	15.0	6.000	68.0568	69.8318	69.1498	68.1580	68.9665	69.0205	68.6951
15	15.0	7.000	70.6726	72.0609	71.4110	70.7627	71.3584	71.2903	71.0561
16	15.0	8.000	73.3896	74.4350	73.8493	73.4632	73.8916	73.7133	73.5923
17	15.0	9.000	76.1919	76.9437	76.4466	76.2478	76.5452	76.2787	76.2761
18	15.0	10.000	79.0669	79.5761	79.1834	79.1062	79.3024	78.9777	79.0822
19	15.0	11.000	82.0049	82.3218	82.0395	82.0299	82.1498	81.8028	81.9883
20	15.0	12.000	84.9979	85.1709	84.9945	85.0118	85.0763	84.7475	84.9744
21	15.0	13.000	88.0395	88.1140	88.0276	88.0455	88.0731	87.8064	88.0231
22	15.0	14.000	91.1242	91.1423	91.1187	91.1257	91.1324	90.9746	91.1187
23	15.0	14.500	92.6814	92.6859	92.6797	92.6818	92.6834	92.5983	92.6799
24	15.0	15.000	94.2478	-	-	94.2478	94.2478	94.2478	94.2478

CONCLUSION

In this paper the several existing formulae have been shown for the reference in order to understand the necessity of accurate formula. Although there are several empirical formulae are exists in practice, the new formula has been derived mathematically from the basic mathematical logic of the ellipse with appropriate illustrations for detailed explanation. A graph drawn for specific values of ‘a’ & ‘b’ also plotted to study the pattern of the curve connecting the extreme points of the perimeter-radius. In the section of "Result & Discussion", the results obtained from the new formula has been analyzed and compared with the other results for the same considered variables 'a' & 'b'. Moreover, finally a new theorem has also been presented.

As per Keplar's first law of planetary motion, all Planets including *Comets* [Eliot. T.S (1994-2011)] and *Asteroids* [Ed Grayzeck (2012)] which are in space rotate in elliptical path as its star is one of the foci of that ellipse. There are many possibilities of hazards and natural calamity due to very high impact will take place when collision of planets and comets or/asteroids which are rotating around our sun having orbit of higher eccentricity are more vulnerable especially to earth. It is one of the natural and solar terrestrial-phenomena happen normally in space. Therefore, this may have application to trace out the path of the comets/ asteroids accurately in order to take precautionary and preventive measures before its collision on earth. Therefore, the new formula which is more accurate and simple is derived mathematically and may have applications in the field of astronomical research, studies and observations. The theorem, which has been defined in this paper is also useful for those doing research or further study in the field of conics and *Euclidean geometry* [Eric Weisstein (2002)] since this is also one of the important properties of an ellipse. It is an advancement in Geometry and *Menstruation* [Eric Weisstein (2002)] related to ellipse.

NOMENCLATURES

a = Semi-major axis
b = Semi-minor axis
e = Eccentricity of ellipse e ≤ 1.0
P = Perimeter or Perimeter of ellipse
π = A transcendental number ≈ 3.141 592 653 589 79…
h = $\left(\frac{a-b}{a+b}\right)^2$
y = $\frac{\ln(2)}{\ln\left(\frac{\pi}{2}\right)}$
H_p = Holder mean of the principal radii = $\left[\frac{(a^p+b^p)}{2}\right]^{\frac{1}{p}}$
p = $\frac{3}{2}$
H_{-p} = $\frac{ab}{H_p}$
f = h^6
δ = $\left(\frac{a^2-b^2}{a^2+b^2}\right)^2$
r_p = Perimeter-radius
k = $\frac{1}{512}$

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