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***Research Article***

# FRACTIONAL INTEGRATION OF THE ALEPH FUNCTION V/A PATHWAY OPERATOR

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**ABSTRACT**
Pathway parameter and pathway model are introduced by Mathai (2005). The object of this paper is to derive the pathway model associated with the Aleph function
When $\alpha \to 1$ , then main results can be reduced to Laplace transform. The results are derived in a closed and compact form. Some special cases are also discussed. A result given by Nair (2009) is deduced as a special case

**Mathematics Subject Classification**: 33C60, 26A33

***Key Words and Phrases*** *:- ℵ function , Pathway model, probability density function, fractional integrals, I function , Fox's H-function.*

**INTRODUCTION**
Let $f(x) \in L(a,b)$, $\alpha \in C$, $\Re(\alpha) > 0$, then left sided Riemann-Liouville fractional integral operator as defined as

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \qquad \ldots(1)$$

Where $\Re(\alpha) > a$.

Saigo hypergeometric fractional integral is defined as [See Kiryakova (1994), Saigo (1978). Samko, Kilbas & Marichev (1953)]

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta ; \alpha ; 1-\frac{t}{x}\right) f(t)\, dt \quad \Re(\alpha)>0 \quad \ldots(2)$$

Two results associated with The Mathai pathway model are investigated in this paper,

**DEFINITION 1** Let $f(x) \in L(a,b)$, $\eta \in C$, $\Re(\eta) > 0$, $a > 0$ and let us take a "pathway parameter" $\alpha < 1$. Then the pathway fractional integration operator is defined by Mathai (2005)

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\, dt \qquad \ldots(3)$$

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For more details which is an extension of the operator defined by (1) of pathway model and its application Mathai and Haubold (2007), (2008). For $\Re(\alpha)$ , the pathway model for scalar random variables is represented by the following probability density function (p.d.f.)

,

$$f(x) = c\,|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \qquad \ldots(4)$$

$-\infty < x < \infty, \quad \delta > 0, \ \beta \geq 0, \ 1 - a(1-\alpha)|x|^{\delta} > 0, \ \gamma > 0,$ where c is the normalizing constant and α is the pathway parameter. For real α , the normalizing constant is as follows :

$$c = \frac{1}{2}\frac{\delta\,[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left[\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right]}{\Gamma\left[\frac{\gamma}{\delta}\right]\Gamma\left[\frac{\beta}{1-\alpha} + 1\right]}, \text{ for } \alpha < 1. \qquad \ldots(5)$$

$$= \frac{1}{2}\frac{\delta\left[a(\alpha-1)\right]^{\frac{\gamma}{\delta}}\,\Gamma\left[\frac{\beta}{\alpha-1}\right]}{\Gamma\left[\frac{\gamma}{\delta}\right]\Gamma\left[\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right]}, \ for \ \frac{1}{1-\alpha} - \frac{\gamma}{\delta} > 0, \ \alpha > 1. \qquad \ldots(6)$$

$$= \frac{1}{2}\frac{\delta(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)} \text{ for } \alpha \to 1. \qquad \ldots(7)$$

It may be observed that for $\alpha < 1$ it is a finite range density with $1 - a(1-\alpha)|x|^{\delta} > 0$ and (4) remains in the extended generalized type-1 beta family. The pathway density in (4) , for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$ , writing $1-\alpha = -(\alpha-1)$ we have

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta}\int_{0}^{\left[\frac{x}{-a(\alpha-1)}\right]}\left[1 + \frac{a(\alpha-1)t}{x}\right]^{\frac{\eta}{-(\alpha-1)}} f(t)\,dt \qquad \ldots(8)$$

$$f(x) = c\,|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}} \qquad \ldots(9)$$

$-\infty < x < \infty, \quad \delta > 0, \ \beta \geq 0, \ \alpha > 1,$ which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density the Student-t density. the Cauchy density.

Here we consider the case of pathway parameter for $\alpha < 1$. For $\alpha \to 1$ both (4) and (9) take the exponential form, since

$$\lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\eta}{1-\alpha}} = \lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\eta}{\alpha-1}}$$

$$= \lim_{\alpha \to 1} c\,|x|^{\gamma-1} e^{-a\eta|x|^{\delta}} \qquad \ldots (10)$$

When $\alpha = 0$, $a = 1$, $\eta$ is replaced by $\eta - 1$ in (3), it yields

$$(I_{0+}^{\eta} f)(x) = \frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1} f(t)\,dt \qquad . . . (11)$$

and reduces to the left-sided Riemann-Liouville fractional integral in (1)

**LEMMA 1**. Let $\eta \in C, \Re(\eta) > 0$, $\beta \in C$ *and* $\alpha < 1$. *If* $\Re(\beta) > 0$, $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$, then there holds the result:

$$P_{0+}^{(\eta,\alpha)}\left[x^{\beta-1}\right] = \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\,\Gamma\left[1 + \frac{\eta}{1-\alpha}\right]}{\Gamma\left[\frac{\eta}{1-\alpha} + \beta + 1\right]} \qquad \ldots (12)$$

Aleph-function is discussed to Südland, Baumann & Nonnenmacher (1998), (2001) and Saxena & Pogány (2010) , this function is defined in the following manner in terms of the Mellin-Barnes type integrals: [See Srivastava & Tomovski (2004)]

$$\aleph[Z] = \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[Z \left| \begin{matrix} (a_j, A_j)_{1,n}, \ldots, [\tau_i(a_j, A_j)]_{n+1,P_i} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_i(b_j, B_j)]_{m+1,q_i} \end{matrix} \right.\right] = \frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)\, z^{-s}\, ds \qquad \ldots (13)$$

for all $z \neq 0$, $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right).\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right).\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)}\quad ...\ (14)$$

The integration path $L=L_{i\gamma\infty},\gamma\in\Re$ extends from $\gamma-i\infty$ *to* $\gamma+i\infty$, and is such that the poles of $\Gamma\left(1-a_j-A_js\right),\ j=\overline{1,n}$ do not coincide with the poles of $\Gamma\left(b_j+B_js\right),\ j=\overline{1,m}$. The parameters $p_i,q_i$ are non-negative integers satisfying the condition $0\le n\le p_i,\quad 1\le m\le q_i,\ \tau_i>0$ for $i=\overline{1,r}$. The parameters $A_j,B_j,A_{ji},B_{ji}>0$ *and* $a_j,b_j,a_{ji},b_{ji}\in C$. An empty product in (14) is interpreted as unity. The existence conditions for the defining integral (13) are given below

$$\varphi_l>0,\quad |\arg(z)|<\frac{\pi}{2}\varphi_l\quad l=\overline{1,r};\qquad (15)$$

$$\varphi_l\ge0,\quad |\arg(z)|<\frac{\pi}{2}\varphi_l\ \ \textit{and}\ \ \Re\{\zeta_l\}+1<0,\qquad (16)$$

where $\varphi_l=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\tau_l\left(\sum_{j=n+1}^{p_l}A_{jl}+\sum_{j=m+1}^{q_l}B_{jl}\right)$ (17)

$$\zeta_l=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+\tau_l\left(\sum_{j=m+1}^{q_l}b_{jl}-\sum_{j=n+1}^{p_l}a_{jl}\right)+\frac{1}{2}\left(p_l-q_l\right)\quad l=\overline{1,r}.\qquad (18)$$

For $\tau_1=\tau_2=...=\tau_r=1$, in (13) the definition of following I-function defined by Saxena (1982) is obtained [Saxena & Pogány (2010), p.982, eqn.(7)],

$$I^{m,n}_{p_i,q_i;r}[z]=\aleph^{m,n}_{p_i,q_i,1;r}[z]=\aleph^{m,n}_{p_i,q_i,1;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},...,(a_j,A_j)_{n+1,p_i}\\(b_j,B_j)_{1,m},...,(b_j,B_j)_{m+1,q_i}\end{matrix}\right.\right]:=\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,1;r}\left(s\right)z^{-s}\,ds\qquad (19)$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}\left(s\right)$ is defined in (14). The existence conditions for the integral in (19) are the same as given in (15) - (18) with $\tau_i=1,\ i=\overline{1,r}$ [(see Saxena, Ram & Chauhan (2002)]
If we set r = 1, then (19) reduces to the familiar H- function [Saxena & Pogány (2010), p.982, eqn.(8)],

$$H[Z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}[Z]=\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[Z\left|\begin{matrix}\left(a_p,A_p\right)\\\left(b_q,B_q\right)\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)z^{-s}\,ds\ ,\qquad ...\ (20)$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;1}(s)$ is defined in (14) ,which itself is a generalization of Meijer's G-function [Erdélyi, Magnus, Oberhettinger & Tricomi.(1953) p. 207] to which it reduces for $A_1=...=A_p=1=B_1=..=B_q$. A detailed and comprehensive account of the H-function is available from the monographs written by Srivastava, Gupta & Goyal (1982) , Kilbas & Saigo (2004) and Mathai, Saxena & Haubold (2010).

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In what follow, the Aleph function will be represented by the contracted notations $\aleph^{m,n}_{p_i,q_i,\tau_i;r}[Z]$ or $\aleph[Z]$.

**Note** : A detailed and comprehensive account of the H-function is available from the monographs Erdélyi, Magnus, Oberhettinger & Tricomi.(1953).

**Theorem 1**. Let $\eta,\rho \in C$, $\Re(\beta) > 0$, $\Re\left(1+\frac{\eta}{1-\alpha}\right) > 0$, $\Re(\rho) > 0$ and $\alpha < 1, b \in R$. Then for the pathway fractional integral $P_{0+}^{(\eta,\alpha)}$ the following formula holds:

$$\left(P_{0+}^{(\eta,\alpha)}\ t^{\rho-1}\ \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left[b\,t^{\beta}\left|\begin{matrix}(a_j,A_j)_{1,n},\dots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\dots,[\tau_j(b_j,B_j)]_{m+1,q_i}\end{matrix}\right.\right]\right)$$

$$=\frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}\ \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left[\frac{bx^{\beta}}{a(1-\alpha)^{\beta}}\left|\begin{matrix}(1-\rho,\beta),\ (a_j,A_j)_{1,n},\dots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left(-\rho-\frac{\eta}{1-\alpha},\beta\right),\dots,[\tau_j(b_j,B_j)]_{m+1,q_i}\end{matrix}\right.\right]\quad \dots(21)$$

Proof :- Uning equation (3) and (13), we have

$$I = x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]} t^{\rho-1}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}}\frac{1}{2\prod\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\left(bt^{\beta}\right)^{-s}\,ds\,dt$$

$$=x^{\eta}\frac{1}{2\prod\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)(b)^{-s}\,ds\int_0^{\left[\frac{x}{a(1-\alpha)}\right]} t^{\rho-\beta s-1}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}}dt$$

Put $\frac{a(1-\alpha)t}{x} = V$ and interchange the order of integration and evaluate the inner integral by beta function formula, it gives

$$I=x^{\eta}\frac{1}{2\prod\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)(b)^{-s}\,ds\left[\frac{x}{a(1-\alpha)}\right]^{\rho-\beta s}\int_0^1 v^{\rho-\beta s-1}\,[1-v]^{\frac{\eta}{(1-\alpha)}+1-1}\,dv$$

$$= x^{\eta} \frac{1}{2\Pi\omega} \int_L \Omega^{m,n}_{p_i, q_i, \tau_i; r}(s)(b)^{-s}\, ds \left[\frac{x}{a(1-\alpha)}\right]^{\rho-\beta s} \beta\left[(\rho-\beta s)\ ;\ \left(\frac{\eta}{1-\alpha}+1\right)\right]$$

$$= \frac{x^{\eta}\ x^{\rho}}{[a(1-\alpha)]^{\rho}} \frac{1}{2\Pi\omega} \int_L \Omega^{m,n}_{p_i, q_i, \tau_i; r}(s)(b)^{-s}\, ds \left[\frac{x}{a(1-\alpha)}\right]^{-\beta s} \frac{\Gamma(\rho-\beta s)\,\Gamma\left(\frac{\eta}{1-\alpha}+1\right)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho-\beta s\right)}$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} \frac{1}{2\Pi\omega} \int_L \Omega^{m,n}_{p_i, q_i, \tau_i; r}(s) \left[\frac{b\,x^{\beta}}{a(1-\alpha)^{\beta}}\right]^{-s} \frac{\Gamma(\rho-\beta s)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho-\beta s\right)}\, ds$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} \aleph^{m,n+1}_{p_i+1, q_i+1, \tau_i; r}\left[\frac{bx^{\beta}}{a(1-\alpha)^{\beta}}\ \middle|\ \begin{matrix} (1-\rho,\beta),\ (a_j, A_j)_{1,n}, \ldots, \left[\tau_j (a_j, A_j)\right]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \left(-\rho-\frac{\eta}{1-\alpha}, \beta\right), \ldots, \left[\tau_j (b_j, B_j)\right]_{m+1, q_i} \end{matrix}\right]$$

**Corollary 1.1** Let $\eta, \rho \in C, \Re(\beta) > 0, \Re\left(1+\frac{\eta}{1-\alpha}\right) > 0, \Re(\rho) > 0, \alpha < 1, b \in R$ and $\tau_1 = \ldots = \tau_r = 1$ then theorem (1) reduces to

$$\left(P^{(\eta,\alpha)}_{0+}\ t^{\rho-1} \aleph^{m,n}_{p_i, q_i, 1; r}\left[b\,t^{\beta}\ \middle|\ \begin{matrix} (a_j, A_j)_{1,n}, \ldots, \left[(a_j, A_j)\right]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \ldots, \left[(b_j, B_j)\right]_{m+1, q_i} \end{matrix}\right]\right)$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} I^{m,n+1}_{p_i+1, q_i+1; r}\left[\frac{bx^{\beta}}{a(1-\alpha)^{\beta}}\ \middle|\ \begin{matrix} (1-\rho,\beta), (a_j, A_j)_{1,n}, \ldots, \left[(a_j, A_j)\right]_{n+1, P_i} \\ (b_j, B_j)_{1,m}, \left(-\rho-\frac{\eta}{1-\alpha}, \beta\right), \ldots, \left[1, (b_j, B_j)\right]_{m+1, q_i} \end{matrix}\right] \quad \ldots (22)$$

where $I(\cdot)$ is the I-function defined in Saxena (1982).

**Corollary 1.2** Let $\eta,\rho \in C, \Re(\beta)>0, \Re\left(1+\frac{\eta}{1-\alpha}\right)>0, \Re(\rho)>0, \alpha<1, b\in R$ and $r = 1$ then corollary (1.1) redues to [8, p. 242]

$$\left( P_{0+}^{(\eta,\alpha)}\, t^{\rho-1} \aleph_{p_i, q_i, 1, 1}^{m,n}\left[ b\,t^{\beta} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,n}, \ldots, [1, (a_j, A_j)]_{n+1, P_i} \\ (b_j, B_j)_{1,m}, \ldots, [1, (b_j, B_j)]_{m+1, q_i} \end{matrix} \right] \right)$$

$$= \frac{x^{\eta+\rho}\, \Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} H_{p+1,q+1}^{m,n+1}\left[ \frac{bx^{\beta}}{a(1-\alpha)^{\beta}} \,\middle|\, \begin{matrix} (a_p, A_p), (1-\rho, \beta) \\ (b_q, B_q), \left(-\rho-\frac{\eta}{1-\alpha}, \beta\right) \end{matrix} \right] \qquad \ldots(23)$$

where $H(\cdot)$ is the H-function defined in Mathai, Saxena & Haubold (2010).

**Corollary 1.3** Let $\eta,\rho \in C, \Re(\beta)>0, \Re\left(1+\frac{\eta}{1-\alpha}\right)>0, \Re(\rho)>0, \alpha<1, b\in R$, $\alpha=0$, $a=1$ and $\eta=\eta-1$ then theorem (1) reduces to

$$\left( I_{0+}^{\eta}\, t^{\rho-1} \aleph_{p_i, q_i, \tau_i, r}^{m,n}\left[ b\,t^{\beta} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,n}, \ldots, [\tau_j (a_j, A_j)]_{n+1, P_i} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_j (b_j, B_j)]_{m+1, q_i} \end{matrix} \right] \right)$$

$$= \frac{x^{\rho+\eta-1}\, \Gamma\left(1+\frac{\eta-1}{1-0}\right)}{[1(1-0)]^{\rho}} \aleph_{p_i+1, q_i+1, \tau_i; r}^{m,n+1}\left[ \frac{bx^{\beta}}{1(1-0)^{\beta}} \,\middle|\, \begin{matrix} (1-\rho, \beta), (a_j, A_j)_{1,n}, \ldots, [\tau_j (a_j, A_j)]_{n+1, P_i} \\ (b_j, B_j)_{1,m}, \left(-\rho-\frac{\eta-1}{1-0}, \beta\right), \ldots, [\tau_j (b_j, B_j)]_{m+1, q_i} \end{matrix} \right]$$

$$= x^{\eta+\rho-1}\, \Gamma(\eta)\, \aleph_{p_i+1, q_i+1, \tau_i; r}^{m,n+1}\left[ bx^{\beta} \,\middle|\, \begin{matrix} (1-\rho, \beta), (a_j, A_j)_{1,n}, \ldots, [\tau_i (a_j, A_j)]_{n+1, P_i} \\ (b_j, B_j)_{1,m}, (-\rho-\eta+1, \beta), \ldots, [\tau_i (b_j, B_j)]_{m+1, q_i} \end{matrix} \right] \qquad \ldots (24)$$

**Theorem 2**. Let $\eta,\rho \in C, \Re(\beta)>0, \Re\left(1-\frac{\eta}{\alpha-1}\right)>0, \Re(\rho)>0$ and $\alpha>1, b\in R$. Then for the pathway fractional integral $P_{0+}^{(\eta,\alpha)}$ the following formula holds:

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$$\left( P_{0+}^{(\eta,\alpha)} \; t^{\rho-1} \aleph_{p_i,q_i,\tau_i,r}^{m,n} \left[ b t^{\beta} \left| \begin{matrix} (a_j, A_j)_{1,n}, \ldots, [\tau_j (a_j, A_j)]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, \ldots, [\tau_j (b_j, B_j)]_{m+1,q_i} \end{matrix} \right. \right] \right)$$

$$= \frac{x^{\eta+\rho} \, \Gamma\left(1 - \frac{\eta}{\alpha-1}\right)}{[-a(\alpha-1)]^{\rho}} \aleph_{p_i+1,q_i+1,\tau_i;r}^{m,n+1} \left[ \frac{b x^{\beta}}{-a(\alpha-1)^{\beta}} \left| \begin{matrix} (1-\rho, \beta), (a_j, A_j)_{1,n}, \ldots, [\tau_j (a_j, A_j)]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, \left(\frac{\eta}{\alpha-1} - \rho, \beta\right), \ldots, [\tau_j (b_j, B_j)]_{m+1,q_i} \end{matrix} \right. \right] \ldots (25)$$

Proof :- Uning equation (8) and (13), we have

$$\Omega = x^{\eta} \int_0^{\left[\frac{x}{-a(\alpha-1)}\right]} t^{\rho-1} \left[1 + \frac{a(\alpha-1)t}{x}\right]^{\frac{\eta}{-(\alpha-1)}} \frac{1}{2\prod \omega} \int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s) \left(b t^{\beta}\right)^{-s} ds \; dt$$

$$= x^{\eta} \frac{1}{2\prod \omega} \int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)(b)^{-s} \, ds \int_0^{\left[\frac{x}{-a(\alpha-1)}\right]} t^{\rho-1-\beta s} \left[1 + \frac{a(\alpha-1)t}{x}\right]^{\frac{\eta}{-(\alpha-1)}} dt$$

Put $\frac{-a(\alpha-1)t}{x} = V$ and interchange the order of integration and evaluating the inner intergral by beta function formula, it gives

$$\Omega = x^{\eta} \frac{1}{2\prod \omega} \int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)(b)^{-s} \, ds \left[\frac{x}{-a(\alpha-1)}\right]^{\rho-\beta s} \beta\left[(\rho - \beta s) \; ; \left(\frac{\eta}{-(\alpha-1)} + 1\right)\right]$$

$$= \frac{x^{\eta} \; x^{\rho}}{[-a(\alpha-1)]^{\rho}} \frac{1}{2\prod \omega} \int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)(b)^{-s} \, ds \left[\frac{x}{-a(\alpha-1)}\right]^{-\beta s} \frac{\Gamma(\rho-\beta s)\,\Gamma\left(\frac{\eta}{-(\alpha-1)} + 1\right)}{\Gamma\left(1 - \frac{\eta}{\alpha-1} + \rho - \beta s\right)}$$

$$= \frac{x^{\eta+\rho} \, \Gamma\left(1 - \frac{\eta}{\alpha-1}\right)}{[-a(\alpha-1)]^{\rho}} \frac{1}{2\prod \omega} \int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s) \left[\frac{b x^{\beta}}{-a(\alpha-1)^{\beta}}\right]^{-s} \frac{\Gamma(\rho-\beta s)}{\Gamma\left(1 - \frac{\eta}{\alpha-1} + \rho - \beta s\right)} ds$$

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$$=\frac{x^{\eta+\rho}\,\Gamma\left(1-\frac{\eta}{\alpha-1}\right)}{\left[-a(\alpha-1)\right]^{\rho}}\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i;r}\left[\frac{bx^{\beta}}{-a(\alpha-1)^{\beta}}\left|\begin{matrix}(1-\rho,\beta),\left(a_j,A_j\right)_{1,n},\ldots,\left[\tau_j\left(a_j,A_j\right)\right]_{n+1,p_i}\\ \left(b_j,B_j\right)_{1,m},\left(\frac{\eta}{\alpha-1}-\rho,\beta\right),\ldots,\left[\tau_j\left(b_j,B_j\right)\right]_{m+1,q_i}\end{matrix}\right.\right]$$

**Corollary 2.1** Let $\eta,\rho\in C,\Re(\beta)>0,\ \Re\left(1-\frac{\eta}{\alpha-1}\right)>0,\ \Re(\rho)>0,\ \alpha>1, b\in R$ and $\tau_1=\ldots=\tau_r=1$ then theorem (2) yields

$$\left(P^{(\eta,\alpha)}_{0+}\ t^{\rho-1}\aleph^{m,n}_{p_i,q_i,1,r}\left[bt^{\beta}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\ldots,\left[\left(a_j,A_j\right)\right]_{n+1,P_i}\\ \left(b_j,B_j\right)_{1,m},\ldots,\left[\left(b_j,B_j\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right)$$

$$=\frac{x^{\eta+\rho}\,\Gamma\left(1-\frac{\eta}{\alpha-1}\right)}{\left[-a(\alpha-1)\right]^{\rho}}I^{m,n+1}_{p_i+1,q_i+1;r}\left[\frac{bx^{\beta}}{-a(\alpha-1)^{\beta}}\left|\begin{matrix}(1-\rho,\beta),\left(a_j,A_j\right)_{1,n},\ldots,\left[\left(a_j,A_j\right)\right]_{n+1,P_i}\\ \left(b_j,B_j\right)_{1,m},\left(\frac{\eta}{\alpha-1}-\rho,\beta\right),\ldots,\left[\left(b_j,B_j\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\quad\ldots(26)$$

where I (·) is the I-function defined in Saxena (1982).

**Corollary 2.2** Let $\eta,\rho\in C,\Re(\beta)>0,\Re\left(1-\frac{\eta}{\alpha-1}\right)>0,\Re(\rho)>0,\alpha<1,b\in R$ and $\tau_1=\ldots=\tau_r=1$ and we set r = 1 then corollary (1.1) reduce to

$$\left(P^{(\eta,\alpha)}_{0+}\ t^{\rho-1}\aleph^{m,n}_{p_i,q_i,1,1}\left[\frac{bx^{\beta}}{-a(\alpha-1)^{\beta}}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\ldots,\left[\left(a_j,A_j\right)\right]_{n+1,P_i}\\ \left(b_j,B_j\right)_{1,m},\ldots,\left[\left(b_j,B_j\right)\right]_{m+1,q_i}\end{matrix}\right.\right]\right)$$

$$=\frac{x^{\eta+\rho}\,\Gamma\left(1-\frac{\eta}{\alpha-1}\right)}{\left[-a(\alpha-1)\right]^{\rho}}H^{m,n+1}_{p+1,q+1}\left[\frac{bx^{\beta}}{-a(\alpha-1)^{\beta}}\left|\begin{matrix}\left(a_p,A_p\right)\ ,(1-\rho,\beta)\\ \left(b_q,B_q\right)\ ,\left(\frac{\eta}{\alpha-1}+\rho,\beta\right)\end{matrix}\right.\right]\quad\ldots(27)$$

where H (·) is the H-function defined in Mathai, Saxena & Haubold (2010).

**REFERENCES**

**Erdélyi A, Magnus W, Oberhettinger F and Tricomi. FG (1953).** Higher Transcendental Functions, Vol. 3, McGraw-Hill, New York,

**Kilbas AA and Saigo M (2004).** H-Transforms: Theory and Application, Chapman & Hall/CRC press, Boca Raton, New York.

**Kiryakova V (1994).** Generalized Fractional Calculus and Applications, Longman Science and Technical, Harlow, U.K. & John Wiley & Sons, New York.

**Mathai AM (2005).** A pathway to matrix-variate gamma and normal densities, Linear Algebra and Its Applications, (396) 317-328.

**Mathai AM and Haubold HJ (2008).** On generalized distributions and Pathway, Physics Letters A 372 2109-2113.

**Mathai AM and Haubold HJ (2007).** Pathway model, superstatistics, Tsallis Statistics and generalized measure of entropy, Physica A: Statistical Mechanics and its Applications 375 110-122.

**Mathai AM, Saxena RK and Haubold HJ (2010).** The H-function: Theory and Application, Springer, New York.

**Nair SS (2009).** Pathway Fractional Integration Operator, Fractional Calculus and Applied Analysis, Vol. 12 (3) 237-251.

**Saigo M (1978).** A Remark on integral operators involving the Gauss hypergeometric functions, Mathematical Reports of College of General Education Kyushu University (11) 135-143.

**Samko SG, Kilbas AA and Marichev OI (1953).** Fractional Integrals and Derivatives, Theory and Applications. Gordon and Breach, New York.

**Saxena RK, Ram J and Chauhan AR (2002).** Fractional integration of the product of the I-function and Appell function $F_3$, Vijnana Parishad Anusandhan Patrika 45 (4) 345-371.

**Saxena RK and Pogány TK (2010).** Mathieu-type series for the $\aleph$-function occurring in Fokker-Planck Equation, European Journal of Pure and Applied Mathematics Vol. 3 (6) 980-988.

**Saxena VP (1982).** Formal solution of certain new pair of dual integral equations involving H-function, Proceedings of the National Academy of Sciences Sect A , 51 366-375.

**Srivastava HM, Gupta KC and Goyal SP (1982).** The H-function of one and two Variables with Applications , South Asian Publishers, New Delhi.

**Srivastava HM and Tomovski Z (2004).** Some problems and solutions involving Mathieu's series and its generalizations, Journal of Inequalities in Pure and Applied Mathematics 5 (2) 1-13.

**Südland N, Baumann B and Nonnenmacher TF(1998).** Open problem: who knows about the Aleph $(\aleph)$-functions ? , Fractional Calculus and Applied Analysis 1(4) 401-402.

**Südland N, Baumann B and Nonnenmacher TF (2001).** Fractional driftless Fokker-Planck equation with power law diffusion coefficients, Edited by Gangha VG, Mayr EW, and Vorozhtsov WG, (Computer Algebra in Scientific Computing, Konstanz, Springer, Berlin ) 513-525.