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Research Article

ON SOME POLYNOMIAL INEQUALITIES NOT VANISHING IN A DISK

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ABSTRACT

If $P(z)$ is a polynomial of degree n , having no zeros in the unit disk ,then for all $\alpha , \beta \in C$ with $|\alpha| \leq 1, |\beta| \leq 1$, it is known that

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z)\right| \leq \frac{1}{2}\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| |z|^n +\right.$$

$1-\alpha+\beta R+12n \ -\alpha$ max$z=1Pz$, for R ≥ 1 and $z \geq 1$.

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INTRODUCTION

Let $P(z)$ be a polynomial of degree atmost n, then according to a famous result known as Bernsteins inequality (for refrence ,see[1994,p.531]or [1941]),

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{1}$$

Whereas concerning the maximum modulus of $P(z)$ on a large circle $|z|$=R > 1 , we have

$$\max_{|z|=R>1}|P(z)| \leq R^n \max_{|z|=1}|P(z)| \tag{2}$$

(for reference see [1994,p.442] or [1925,vol.1,p.137]).

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$,then inequalities (1) and (2) can be sharpened . In fact , $P(z) \neq 0$ in $|z| < 1$, then (1) and (2)can be respectively replaced by

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)| \tag{3}$$

and

$$\max_{|z|=R>1}|P(z)| \leq \frac{R^n+1}{2} \max_{|z|=1}|P(z)| \tag{4}$$

Inequality (3) was conjectured by Erd*Ös* and later verified by Lax [1944](see also [1980]), whereas Ankeny and Rivlin [1955] used (3) to prove (4) .

Research Article

Recently both the inequalities (3) and (4) were further improved by Jain [1997], who proved that if $P(z) \neq \max_{|z|=1}|P'(z)|$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2}\left[\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right]\max_{|z|=1}|P(z)|, \tag{5}$$

And $\left|P(Rz) + \beta\left\{\left(\frac{R+1}{2}\right)^n\right\}P(z)\right| \leq \frac{1}{2}\left[\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right]\max_{|z|=1}|P(z)|$,
for R ≥ 1 and
$|z| = 1$ (6)

More recently Abdul Aziz and Nisar Ahmad Rather [2004] have investigated the dependence of

$$\max_{|z|=1}|P(Rz) - \alpha P(z)|$$

on $\max_{|z|=1}|P(z)|$ for every real or complex number α and R≥ 1 . As a compact generalisation of inequalities (1) and (2), they have shown that if $P(z)$ is a polynomial of degree n ,then for all real or complex number α with $|\alpha| \leq 1$ and R≥ 1

$$|P(Rz) - \alpha P(z)| \leq |R^n - \alpha|\,|z|^n \max_{|z|=1}|P(z)| \tag{7}$$

for $|z| \geq 1$. This results is sharp and equality holds for $P(z) = \lambda z^n$, $\lambda \neq 0$. Inequality (1) can be obtained from inequality (7) by dividing the two sides of (7) by R-1 and taking limit R→ 1, with $\alpha = 1$.For $\alpha = 0$, inequality (7) reduces to (2) .

As a corresponding compact generalisation of inequality (3) and (4), Abdul Aziz and Nisar Ahmad Rather [2004] have already shown that if $P(z)$ is a polynomial of degree n , which has no zeros in $|z| < 1$, ,then for all $\alpha \in C\ or\ R$ with $|\alpha| \leq 1$ and R≥ 1 ,

$$|P(Rz) - \alpha P(z)| \leq \frac{1}{2}\left[|R^n - \alpha||z|^n + |1-\alpha|\right]\max_{|z|=1}|P(z)|, \text{ for } |z| \geq 1. \tag{8}$$

This result is sharp and equality holds for $P(z) = z^n + 1$.

Abdul Aziz and Nisar Ahmad Rather [2004] have also proved the following results:

Theorem A. If $P(z)$ is a polynomial of degree n , then for all real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right| + \left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}Q(z)\right|$$
$$\leq \left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||z|^n + \left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right]\max_{|z|=1}|P(z)|, \text{ for}$$

R ≥ 1 and $|z| \geq 1$. where $Q(z) = z^n\,\overline{p(\frac{1}{\bar{z}})}$. This results is sharp and equality holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

Theorem B. If $P(z)$ is a polynomial of degree n , which does not vanish in $|z| < 1\ then\ for\ all$ real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z)\right| \leq \frac{1}{2}\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| |z|^n +\right.$

1−α+βR+12n −α maxz=1Pz, for R ≥1 and z≥1.

This result is sharp and equality holds for $P(z) = z^n + 1$.

Theorem. If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| < 1$ $then\ for\ all$ real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z)\right| \leq \frac{1}{2}\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| |z|^n +\right.$

1−α+βR+12n −α maxz=1Pz− 12 Rn−α+βR+12n −α zn−1−α+βR+12n −α minz=1Pz ,
for R ≥ 1 and $|z| \geq 1$. (9)

This result is sharp and equality holds for $P(z) = z^n + 1$.

Remark1. If we take $\alpha = 0$, in theorem (1) , we get refinement of inequality (6) whereas refinement of inequality (8) follows by taking β =0 in inequality (9) .For $\alpha = \beta = 0$, inequality (9) reduces to refinement of inequality (4).

The next corollary which is obtained by taking $\alpha = 1$ refinement of corollary 1, for polynomials not vanishing in the unit disk due to Aziz and Nisar ahmad rather [2004].

Corollary1. If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| < 1$ $then\ for\ all$ real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$\left|P(Rz) - P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - 1\right\} P(z)\right| \leq \frac{1}{2}\left[\left|R^n - 1 + \beta\left\{\left(\frac{R+1}{2}\right)^n - 1\right\}\right| |z|^n +\right.$

βR+12n −1 maxz=1Pz− 12 Rn−1+βR+12n −1 zn−βR+12n −1 minz=1Pz , For R ≥1 and $|z| \geq 1$.The result is sharp and equality holds for $P(z) = z^n + 1$. (10)

Remark2. Dividing the two sides of (10) by R-1 and letting R$\rightarrow 1$, we obtain ,in particular , refinement of (3).

Lemma1. If $F(z)$ is a polynomial of degree n, which does vanish in $|z| \leq 1$ $and\ P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \quad \text{For} \quad |z| = 1$$

$then\ for\ all$ real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$$\left|P(Rz)- \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z)\right| \leq \left|F(Rz)- \alpha F(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\} F(z)\right|$$

for $|z| \geq 1$.This lemma is due to Abdul Aziz and Nisar ahmad rather.

Lemma2. If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| \leq 1$ $then\ for\ all$ real or complex $\alpha\ and\ \beta$ with $|\beta| \leq 1$, $|\alpha| \leq 1$ and R≥ 1 ,

$$\left|P(Rz)- \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z)\right| \leq \left|Q(Rz)- \alpha Q(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - \right.\right.$$

$\alpha\ Qz\,Rn{-}\alpha{+}\beta R{+}12n\ {-}\alpha\ \ zn{-}1{-}\alpha{+}\beta R{+}12n\ {-}\alpha\ \ \min z{=}1Pz$

Where $Q(z) = z^n\ \overline{p(\frac{1}{\bar{z}})}$ for $|z| \geq 1$.

Proof of lemma 2. $\min_{|z|=1}|P(z)| = m$ which implies $m \leq |P(z)|$ for $|z| = 1$. $for\ any$ real or complex λ, $|\lambda| \leq 1$,

$$m\left|\lambda\right| \leq |P(z)|$$

Therefore G(z)=P(z)+ λm does not vanish in open unit disk by Rouches theorem. If H(z)= $= z^n \overline{G(\frac{1}{\bar{z}})} = Q(z) + \bar{\lambda}\mathrm{m}z^n$. Then it has all zeros in $|z| \leq 1$,and $|G(z)| = |H(z)|$ on $|z| = 1$. Applying lemma 1 with P(z)=G(z) and F(z)=H(z) we have

$$\left|(P(Rz) + \lambda m(Rz)^n)- \alpha\,(P(z) + \lambda m(z)^n) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\} (P(z) + \lambda m(Rz)^n)\right|$$
$$\leq \left|(Q(Rz) + \overline{\lambda \mathrm{m}})- \alpha(\,Q(z) + \overline{\lambda \mathrm{m}}) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\}(\,Q(z) + \overline{\lambda \mathrm{m}})\right|$$

For $|z| \geq 1$, which implies

$$\left|P(Rz) - \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\} P(z) + \lambda m(z)^n \left[R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right]\right| \leq$$
$$\left|Q(Rz)- \alpha Q(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha|\right\}Q(z)\right| + |\bar{\lambda}|\,\left|1 - \alpha + \right.$$

$\beta R{+}12n\ {-}\alpha\ \ \min z{=}1\ Pz$

For $|z| \geq 1$, further choosing argument of λ, suitably, we shall get

Research Article

$$\left|P(Rz) - \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} P(z)\right| + \left|\lambda m(z)^n \left[R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha| \right\}\right]\right| \leq$$
$$\left|Q(Rz) - \alpha Q(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} Q(z)\right| + |\bar{\lambda}| \left|1 - \alpha +\right.$$

$\beta R+12n \ -\alpha$ min$z=1$ Pz

For $|z| \geq 1$ .Letting $|\lambda| \to 1$, we get desired result.

Proof of theorem: Since by lemma 2 , we have

$$\left|P(Rz) - \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} P(z)\right| \leq \left|Q(Rz) - \alpha Q(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} Q(z)\right| -$$
$$\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha| \right\}\right| |z|^n - \left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha| \right\}\right|\right] \min_{|z|=1}|P(z)| ,$$ for R

≥ 1 and $|z| \geq 1$. Where $Q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$. Adding on both sides

Adding on both sides

$$\left|P(Rz) - \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} P(z)\right|$$

And using theorem(A), we get $\left|P(Rz) - \alpha P(z) + \beta\left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} P(z)\right| \leq \frac{1}{2}\left[\left|R^n - \alpha +\right.\right.$

$\beta R+12n \ -\alpha \ zn+1-\alpha+\beta R+12n \ -\alpha$ max$z=1Pz-12Rn-\alpha+\beta R+12n \ -\alpha \ zn-1-\alpha+\beta R+12n$
$-\alpha$ min$z=1Pz$, for R ≥ 1 and $z \geq 1$.

Hence the result.

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