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Research Article

ON THE LOCATION OF ZEROS OF A POLYNOMIAL

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ABSTRACT
In this paper we prove a result concerning the distribution of the zeros of a polynomial in the complex plane. From our result a variety of interesting results can be deduced by a fairly uniform procedure.

Key Words: *Polynomials, Zeros, Analytic functions.*

Mathematics Subject classification (1991): 30C10, 30C15

INTRODUCTION

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \neq 0$ such that for some $k \geq 1$, $\lambda \geq 1$

$$K\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$
$$K\beta_n \geq \beta_{n-1} \geq .... \geq \beta_1 \geq \beta_0$$

then $P(z)$ has all its zeros in

(1) $$\left|z+(K-1)\right| \leq \frac{\left\{K(\alpha_n+\beta_n)-(\alpha_0+\beta_0)+|\alpha_0|\right\}}{|a_n|}$$

Aziz and Mohammad [1980] extended EnestrÖm-Kakeya Theorem to the class of Analytic functions $P(z)=\sum_{j=0}^{n} a_j z^j$ (not identically zero) with its coefficients a_i satisfying a relation analogous to that of EnestrÖm-Kakeya and proved the following theorem.

Theorem B: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leq t$. If $a_j > 0$ and $a_{j-1} - ta_j \geq 0$, $j = 1,2,3,...$ then $f(z)$ does not vanish in $|z| < t$.

Aziz and Shah [1998] relaxed the hypothesis of Theorem B and proved the following.

Theorem C: If $f(z)=\sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| < t$ such that for some $k \geq 1$

$$ka_0 \geq ta_1 \geq t^2 a_2 \geq ....,$$

then $f(z)$ does not vanish in

$$\left|z-\left(\frac{K-1}{2K-1}\right)t\right| \leq \frac{Kt}{2K-1}.$$

Shah and Liman [2007] proved the following result concerning the location of zeros of analytic function.

Theorem D: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z|<t$. If for some $k\geq 1$,

$$k|a_0|\geq t|a_1|\geq t^2|a_2|\geq \ldots ,$$

and for some β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,\ldots,$$

then $f(z)$ does not vanish in

(2) $$\left|z-\frac{(K-1)t}{M^2-(K-1)^2}\right|<\frac{Mt}{M^2-(K-1)^2},$$

where

$$M=K(\cos\alpha+\sin\alpha)+2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j$$

In this paper we prove the following result which not only generalizes Theorem D but in particular cases reduces to Theorem C and Theorem B. More precisely we prove

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z|\leq t$. If for some $K\geq 1$

$$K|a_0|\geq t|a_1|\geq t^2|a_2|\geq\ldots\geq t^{\lambda}|a_{\lambda}|\leq t^{\lambda+1}|a_{\lambda+1}|\leq\ldots$$

and for some real β

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,\ldots,$$

then $f(z)$ does not vanish in

(3) $$\left|z-\frac{(K-1)t}{M_4^2-(K-1)^2}\right|<\frac{M_4 t}{M_4^2-(K-1)^2},$$

where

$$M_4=\left(K-2t^{\lambda}\left|\frac{a_{\lambda}}{a_0}\right|\right)\cos\alpha+K\sin\alpha+2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j$$

Remark 1.1: If we let $\lambda\to\infty$ in Theorem 1 we get Theorem D. For $\lambda\to\infty$, $\alpha=\beta=0$ Theorem 1 reduces to Theorem C and for $\lambda\to\infty$, $\alpha=\beta=0$, $K=1$ Theorem 1 reduces to Theorem B

We need the following lemma [1968] for the proof of the above Theorem.

Lemma: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some real β

$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ \ j=0,1,2,\ldots n$ then for some $t>0$

$$|ta_j-a_{j-1}|\leq\left(|t||a_j|-|a_{j-1}|\right)\cos\alpha+\left(|t||a_j|+|a_{j-1}|\right)\sin\alpha .$$

Proof of Theorem 1: Since $f(z)$ is analytic function in $|z| \le t$ therefore $\lim_{j\to\infty} a_j z^j = 0$. Now consider the function

$$F(z) = (z-t)\,f(z)$$
$$= (z-t)(a_0 + a_1 z + a_2 z^2 + ... a_\lambda z^\lambda + ...)$$
$$= -ta_0 + (a_0 - ta_1)\,z + (a_1 - ta_2)\,z^2 + ...$$
$$= -ta_0 + a_0 z - Ka_0 z + (Ka_0 - ta_1)\,z + \sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^j$$

(4) $$= -ta_0 + a_0 z - Ka_0 z + G(z), \qquad \text{(say)}$$

where

$$G(z) = (Ka_0 - ta_1)\,z + \sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^j$$

clearly $G(z)$ is analytic , $G(0) = 0$ and $|z| = t$

$$|G(z)| = \left|(Ka_0 - ta_1)\,z + \sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^j\right|$$
$$\le t\left|(Ka_0 - ta_1)\right| + t\sum_{j=2}^{\infty}\left|(a_{j-1} - ta_j)\right|t^{j-1}$$
$$\le t\left(K|a_0| - t|a_1|\right)\cos\alpha + t\left(K|a_0| + t|a_1|\right)\sin\alpha$$
$$+\left(t|a_1| - t^2|a_2|\right)\cos\alpha + t\left(t|a_1| + t^2|a_2|\right)\sin\alpha$$
$$+\left(t^2|a_2| - t^3|a_3|\right)\cos\alpha + t\left(t^2|a_2| + t^3|a_3|\right)\sin\alpha + ... +$$
$$+\left(t^{\lambda-1}|a_{\lambda-1}| - t^\lambda|a_\lambda|\right)\cos\alpha + t\left(t^{\lambda-1}|a_{\lambda-1}| + t^\lambda|a_\lambda|\right)\sin\alpha$$
$$+\left(t^\lambda|a_\lambda| - t^{\lambda+1}|a_{\lambda+1}|\right)\cos\alpha + t\left(t^\lambda|a_\lambda| + t^{\lambda+1}|a_{\lambda+1}|\right)\sin\alpha + ...$$

$$= t\left[\left(K|a_0| - 2t^\lambda|a_\lambda|\right)\cos\alpha + K|a_0|\sin\alpha + 2\sin\alpha\sum_{j=1}^{\infty}|a_j|t^j\right]$$

$$= t|a_0|\left[\left(K - 2t^\lambda\left|\frac{a_\lambda}{a_0}\right|\right)\cos\alpha + K\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\right]$$
$$= t|a_0|M_4 \qquad \text{(say)}$$

where

(5) $$M_4 = \left(K - 2t^\lambda\left|\frac{a_\lambda}{a_0}\right|\right)\cos\alpha + K\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j$$

this implies

$|G(z)| \le t|a_0|M_4|z|$; for $|z| \le t$, by Schwartz's lemma

hence from (4) we have

$$|F(z)| \ge |ta_0 - a_0 z + Ka_0 z| - |G(z)|$$
$$\ge |a_0|\left|(K-1)z + t\right| - = |z||a_0|M_4$$
$$>0$$

if

(6) $$|z|M_4 < \left|(K-1)z + t\right|.$$

Since it is easy to verify that the region defined above is precisely the disk

(7) $$\left[z : \left| z - \frac{(K-1)t}{M_4^2 - (K-1)^2} \right| < \frac{M_4 t}{M_4^2 - (K-1)^2} \right]$$

it follows from (6) that $F(z)$ and hence $f(z)$ does not vanish in the the disk defined by (7). This completes the proof of Theorem 1.

REFERENCES

Shah W.M and Liman A (2007). On Eneström-Kakeya theorem and related Analytic Functions, *Proceedings of the Indian Academy of Sciences (Math Science)* **117**(3) 359-370.

Rather N.A and Ahmad S.S (2007). A remark on the generalization of EnestrÖm -Kakeya theorem. *Journal of Analysis and Computation* **3**(1) 33-41

Aziz A and Shah W.M (1998). On the Zeros of Polynomials and related Analytic functions, *Glasnik Matematicki* **33** 173-184.

Aziz A. and Mohammad Q.G (1980). On the zeros of certain class of polynomial and related analytic functions. *Journal of Mathematical Analysis and Application* **75** 495-502.

Govil N.K and Rahman Q.I (1968). On the Eneström- Kakeya theorem, *Tohoku Mathematical Journal*, **20** 126-136.