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Research Article

A NEW GENERALIZATION OF ERLANG DISTRIBUTION WITH BAYES ESTIMATION

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ABSTRACT
The Erlang distribution is the distribution of sum of exponential variates. The Erlang variate becomes Gamma variate when its shape parameter is an integer (Evans *et al.*, 2000). In the literature, various authors discussed the properties and estimation of Erlong distribution e.g., Harischandra and Rao (1988), Bhattacharyya and Singh (1994), Wiper (1998), Jain (2001), Nair *et al.*, (2003), Suri *et al.*, (2009), Damodaran *et al.*, (2010). In this paper, we propose a new generalization of Erlang distribution then discuss the Bayesian estimation of Erlang distribution using different priors. We illustrate the results using a simulation study as well as by doing real data analysis.

Keywords: *Probability Density Function, Bayes Estimator, Posterior Distribution, Prior, Simulation*

INTRODUCTION
The origin of queuing theory was in 1909, when A.K. Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic (Brockmeyer *et al.*, 1948). The literature on the theory of queues and on the diverse field of its applications has grown tremendously over the years. The analysis for such an Erlangian queue is now folklore in the queuing literature. The Erlang distribution is the distribution of sum of exponential variates.
This distribution can be expressed as waiting time and message length in telephone traffic. If the duration of individual calls are exponentially distributed then the duration of succession of calls is the Erlang distribution. The Erlang variate becomes Gamma variate when its shape parameter is an integer (Evans *et al.*, 2000).
In the literature, we observe that Harischandra and Rao (1988) discussed some problems of classical inference for the Erlangian queue. Bhattacharyya and Singh (1994) obtained Bayes estimator for the Erlangian queue under two prior densities. Wiper (1998) studied for Er/*M*/1 and Er/*M*/*c* queues under Bayesian setup and estimated equilibrium probabilities of the queue size and waiting time distributions using conditional Monte-Carlo simulation methods.
Jain (2001) discussed the problem of the change point for the inter arrival time distribution in the context of exponential families for the Ek/GIc queuing system and obtained Bayes estimates of the posterior probabilities and the positions of change from the Erlang distribution. Nair *et al.*, (2003) studied Erlang distribution as a model for ocean wave periods and obtained different characteristics of this distribution under classical set up.
Suri *et al.*, (2009) used Erlang distribution to design a simulator for time estimation of project management process. Damodaran *et al.*, (2010) obtained the expected time between failure measures. Further, they showed that the predicted failure times are closer to the actual failure times.
In view of the literature available on Erlang distribution and its applications, we try to generalize it from a new proposed model (Bilal and Khan, 2011) and discuss its Bayes estimation.
The probability density function (pdf) of Erlang distribution is given by

$$\varphi(x)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta^{-1}x), 0<x<\infty \qquad (1.1)$$

$= 0$ elsewhere, $\quad \alpha = 1,2,3,\ldots, \beta>0$

Proposed Method of Generalization of Erlang Distribution

We now obtain Erlang distribution (1.1) by a new method as given below:
Suppose F(u) be any non-negative continuous function of u defined in the interval 0<u<t and if α is any given positive real number such that

$$A_{F,\alpha} = \int_0^t u^{\alpha-1} F(u)\,du \qquad (2.1)$$

is finite. Then the function

$$\varphi(x) = x^{\alpha - 1} F(x) / A_{F,\alpha};\ 0 < x < t \qquad (2.2)$$

= 0, elsewhere

is a probability density function (pdf) of X, a continuous random variable.
The rth moment of the distribution about the origin is

$$\mu_r^{'} = E(X^r) = \frac{A_{F,\alpha+r}}{A_{F,\alpha}},\ r = 1, 2, \ldots$$

In particular the mean μ and the variance σ^2 are respectively given by

$$\mu = \mu_1^{'} = \frac{A_{F,\alpha+1}}{A_{F,\alpha}}$$

and

$$\sigma^2 = E(X^2) - \mu^2 = \frac{A_{F,\alpha+2}\, A_{F,\alpha} - (A_{F,\alpha+1})^2}{(A_{F,\alpha})^2}.$$

Taking F(u) = exp(-uβ^{-1}) with u = x β and letting t→∞ in (2.1), it follows that

$$A_{F,\alpha} = \beta^{\alpha} \int_0^{\infty} x^{\alpha-1} e^{-x}\,dx$$

$$= \beta^{\alpha}\Gamma(\alpha). \qquad (2.3)$$

Using (2.3) in (2.2), we get

$$\varphi(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta^{-1}x),\ 0 < x < \infty$$

= 0, elsewhere (2.4)

which is the probability density function (pdf) of Erlang distribution with parameters α and β.

Bayes Estimation of Erlang Distribution

Bayesian statistics is an approach to statistics which formally seeks use of prior information with the data, and Bayes Theorem provides the formal basis for making use of both sources of information in a formal manner. Bayes theorem is stated as
Posterior α Likelihood × Prior

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The prior is the probability of the parameter and represents what was thought before seeing the data. The likelihood is the probability of the data given the parameter and represents the data now available. The posterior represents what is thought given both prior information and the data just seen.

In many practical situations, the information about the shape and scale parameters of the sampling distribution is available in an independent manner.

Therefore, here it is assumed that the parameters α and β are independent a priori and a prior distributions chosen in this paper is Truncated Poisson distribution as a prior for shape parameter and Inverted Gamma distribution as a prior for scale parameter. The loss function considered in this paper is squared error loss function. The squared error loss function for the shape parameter *c* and the scale parameter *b* are defined as

$$L(\hat{\alpha}) = (\hat{\alpha} - \alpha)^2 \tag{3.1}$$

$$L(\hat{\beta}) = (\hat{\beta} - \beta)^2 \tag{3.2}$$

which is symmetric and α , β and $\hat{\alpha}$, $\hat{\beta}$ represent the true and estimated values of the parameters.

Posterior Distributions under Different Informative Priors

The posterior distributions using different informative priors for unknown parameters α (shape) and *β* (scale) are derived in the following subsequent subsections.

The probability density function (pdf) of Erlang distribution is given by

$$\varphi(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta^{-1}x), 0 < x < \infty \tag{3.3}$$

= 0 elsewhere, α = 1,2,3, …, β>0

Let *X*1, *X* 2 , ..., *X n* be a random sample from the Erlang distribution, the likelihood function of the sample observations: **x** : *x*1, *x*2 ,..., *xn* is defined as

$$L(\alpha, \beta; x) = \frac{\prod_{i=1}^{n} X_i^{\alpha-1} \exp(-\beta^{-1}\sum_{i=1}^{n} x_i}{\beta^{n\alpha}(\Gamma(\alpha))^n}, \ \alpha = 1,2,3, \ldots, \beta>0 \tag{3.4}$$

When Shape Parameter *α* ***is Unknown and Scale Parameter*** *β* ***is Known***

We assume prior for shape parameter α the truncated Poisson distribution, given by

$$g_1(c; \theta_1) = \frac{\exp(-\theta_1)\theta_1^{\alpha}}{\Gamma(\alpha+1)(1-\exp(-\theta_1))}, \tag{3.5}$$

α = 1,2,3, …, β>0

By combining the likelihood function and the prior density, the posterior distribution of α given data is

$$g_1(\alpha; \theta_1) = \frac{\dfrac{\theta_1^{\alpha} \exp(\alpha \sum_{i=1}^{n} \ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n \beta^{n\alpha}}}{\sum_{\alpha=1}^{\infty} \dfrac{\theta_1^{\alpha} \exp(\alpha \sum_{i=1}^{n} \ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n \beta^{n\alpha}}}, \tag{3.6}$$

α = 1,2,3, …,

The Bayes estimator under squared error loss function with the prior $g_1(\alpha; \theta_1)$ is given by

$$\hat{\alpha_1}\Big|x = \frac{\sum_{\alpha=1}^{\infty}\left(\frac{\alpha\theta_1^{\alpha}\exp(\alpha\sum_{i=1}^{n}\ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n\beta^{n\alpha}}\right)}{\sum_{\alpha=1}^{\infty}\left(\frac{\theta_1^{\alpha}\exp(\alpha\sum_{i=1}^{n}\ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n\beta^{n\alpha}}\right)} \tag{3.7}$$

The posterior variance of Bayes estimator is given by

$$Var(\hat{\alpha_1}\Big|x) = \frac{\sum_{\alpha=1}^{\infty}\left(\frac{\alpha^2\theta_1^{\alpha}\exp(\alpha\sum_{i=1}^{n}\ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n\beta^{n\alpha}}\right)}{\sum_{\alpha=1}^{\infty}\left(\frac{\theta_1^{\alpha}\exp(\alpha\sum_{i=1}^{n}\ln x_i)}{\Gamma(\alpha+1)(\Gamma(\alpha))^n\beta^{n\alpha}}\right)} - (\hat{C_1}\Big|x)^2 . \tag{3.8}$$

When Scale Parameter β is Unknown and Shape Parameter α is Known

We choose Inverted Gamma distribution, the prior for scale parameter β as given by

$$g_1(\beta;\alpha_1,\beta_1) = \frac{\beta_1^{\alpha_1}\beta^{-(\alpha_1+1)}\exp(-\beta^{-1}\beta_1)}{\Gamma(\alpha_1)}, \tag{3.9}$$

$\beta>0,\ (\alpha_1 , \beta_1)>0.$

The posterior density after combining likelihood and prior is given by

$$g_1(\beta|x) = \frac{\left(\beta_1+\sum_{i=1}^{n}x_i\right)^{\alpha_1+n\alpha}\exp\left\{-\beta^{-1}\left(\beta_1+\sum_{i=1}^{n}x_i\right)\right\}}{\Gamma(\alpha_1+n\alpha)\beta^{(\alpha_1+n\alpha+1)}},\quad \beta>0 \tag{3.10}$$

The Bayes estimator under squared loss function is given by

$$\hat{\beta_1}\Big|x = \frac{\beta_1+\sum_{i=1}^{n}x_i}{\alpha_1+n\alpha-1} \tag{3.11}$$

The posterior variance of Bayes estimator $\hat{\beta_1}\Big|x$ is given by

$$Var(\hat{\beta_1}\Big|x) = \frac{\left(\beta_1+\sum_{i=1}^{n}x_i\right)^2}{(\alpha_1+n\alpha-1)^2(\alpha_1+n\alpha-2)}. \tag{3.12}$$

Simulation Study: In the simulation study, we have chosen n=50 for several values of parameters. The simulation program was written in S-Plus/R software. The results obtained y using simulation study is presented below:

Table 1: Bayes Estimators and their Variances under Truncated Poisson Distribution for n=50

α =1		α =3		α =6		α =9	
$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$
1	1.3744×10^{-13}	3.00001	0.00005	5.9586	0.0436	8.7129	0.21342
1	2.7377×10^{-13}	3.00005	0.00008	5.9818	0.0257	8.8304	0.1503
1	4.1142×10^{-13}	3.00010	0.00011	5.9813	0.0262	8.8764	0.1239
1	5.4810×10^{-13}	3.00014	0.00015	5.9867	0.0254	8.8956	0.1078
1	6.8334×10^{-13}	3.00018	0.00017	6.0013	0.0241	8.9284	0.0995
1	8.2256×10^{-13}	3.00022	0.00021	6.0045	0.0164	8.9674	0.0937
1	9.5788×10^{-13}	3.00024	0.00025	6.0069	0.0250	8.9787	0.0912
1	1.0879×10^{-13}	3.00029	0.00030	6.0107	0.0215	8.9992	0.0943
1	1.2336×10^{-13}	3.00027	0.00032	6.0129	0.0234	9.0076	0.0911

Table 2: Bayes Estimators and their Variances under Inverted Gamma Distribution for n=50

β=1		β =3		β =6		β =9	
$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$	$\hat{\beta}_1 \mid x$	$V(\hat{\beta}_1 \mid x)$
1.003010	0.00328	3.00624	0.03020	6.00229	0.12047	9.00431	0.27102
1.00290	0.00327	3.00101	0.02996	5.99052	0.12004	8.94231	0.26789
1.00298	0.00325	2.99241	0.02983	5.98657	0.11941	8.91238	0.26683
1.00307	0.00324	2.99012	0.02972	5.98001	0.11873	8.90231	0.26664
1.00306	0.00323	2.98023	0.02968	5.97541	0.11753	8.89474	0.26510
1.00305	0.00322	2.97432	0.02907	5.96231	0.11634	8.89002	0.26001
1.00304	0.00321	2.96326	0.02875	5.94537	0.11562	8.88321	0.25767
1.00302	0.00320	2.96103	0.02858	5.90231	0.11432	8.83762	0.25546
1.00301	0.00319	2.95431	0.02835	5.86735	0.11310	8.80023	0.25241

Read Data Analysis

To check the validity of the model, we consider the survival time (in weeks) for 20 male rats (Lawless, 2003) that were exposed to a high level of radiation. The data is 152, 152, 115, 109, 137, 88, 94, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83 and 69. The three goodness of fit test reveal that

(i) Kolmogrov-Smirnov test: Test statistic: 0.1478 with p-value 0.720.
(ii) Anderson-Darling test: Test statistic: 0.4921.
(iii) Chi-square test: Test statistic: 0.7832 with p-value 0.675.

From above tests, it is evident that the Erlang distribution with parameters α =10 and β =11.29 fits the data set well. It is observed that results from this data analysis echo the same pattern as found in the simulation study. The results obtained are in agreement with the earlier studies.

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