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ON THE NON HOMOGENEOUS HEPTIC EQUATION WITH FIVE UNKNOWNS $(x^2 - y^2)(4x^2 + 4y^2 - 6xy) = 8(X^2 - Y^2)z^5$

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ABSTRACT
The non-homogeneous Diophantine equation of degree seven with five unknowns represented by$(x^2 - y24x2+4y2-6xy=8X2-Y2z5$is analyzed for its non – zero distinct integer solutions. Employing suitable linear transformations and applying the method of factorization, twodifferent patterns of non-zero distinct integer solutions to the heptic equation under consideration are obtained. A few interesting relation between the solutions and special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, m-dimensional figurate numbers, prism numbers and nexus numbers are exhibited.

Keywords: *Non-Homogeneousheptic, Heptic Equation with Five Unknowns, Integral Solutions, Special Numbers*
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INTRODUCTION

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from (Dickson, 1952; Mordell, 1969). The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree atleast three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables (Gopalan *et al.*, 2006; Gopalan *et al.*, 2013; Gopalan, 2000). Cubic equations in three variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research (Gopalan *et al.*, 2010; Gopalan *et al.*, 2006; Gopalan *et al.*, 2008). A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small co-efficients. It seems that much work has not been done in solving higher order Diophantine equations. In (Gopalan and Sangeetha, 2011; Gopalan *et al.*, 2011; Gopalan and Sangeetha, 2010; Gopalan and Janaki, 2010; Gopalan and Vijayashankar, 2010; Gopalan and Sangeetha, 2012; Gopalan *et al.*, 2014) a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation with five variables represented by$(x^2 - y^2)(4x^2 + 4y^2 - 6xy) = 8(X^2 - Y^2)z^5$is considered and in a few interesting relations among the solutions are presented.

Notations

- $t_{m,n}$- Polygonal number of rank n with size m.
- P_n^m- Pyramidal number of rank n with size m.
- $cp_{m,n}$ - Centered polygonal number of rank n with size m.
- CP_n^m - Centered pyramidal number of rank n with size m.
- $P(n)$ - Pronic number of rank n
- $f_{m,s}^n$ - m-dimensional figurate number of rank n with s sides.
- $N_d(n)$ - d-dimensional nexus number of rank n
- $TOH(n)$ - Truncated octahedral number of rank n
- $CD(n)$ - Centered Dodecahedral number of rank n
- $CI(n)$ - Centered Icosahedral number of rank n

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Method of Analysis

The Diophantine equation representing the Heptic equation with five unknowns under consideration is

$$(x^2-y^2)(4x^2+4y^2-6xy)=8(X^2-Y^2)z^5 \qquad (1)$$

Introduction of the linear transformation

$$x=u+v \;;\; y=u-v \;;\; X=2u+v \;;\; Y=2u-v\;; \qquad (2)$$

in (1) leads to

$$u^2+7v^2=8z^5 \qquad (3)$$

Now, we solve (3) through different methods and thus obtain different patterns of solutions to (1)

Pattern - I

Assume $z=z(a,b)=a^2+7b^2$ (4)

Where a and b are non - zero distinct integers

Write 8 as $8=(1+i\sqrt{7})(1-i\sqrt{7})$ (5)

Using (4) & (5) in (3) and applying the method of factorization, define

$$u+i\sqrt{7}v=(1+i\sqrt{7})(a+i\sqrt{7}b)^5$$

Equating the real and imaginary parts, we have

$$u=u(a,b)=a^5-35a^4b-70a^3b^2+490a^2b^3+245ab^4-343b^5$$
$$v=v(a,b)=a^5+5a^4b-70a^3b^2-70a^2b^3+245ab^4+49b^5$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x=x(a,b)=2a^5-30a^4b-140a^3b^2+420a^2b^3+490ab^4-294b^5$$
$$y=y(a,b)=-40a^4b+560a^2b^3-392b^5$$
$$z=z(a,b)=a^2+7b^2$$
$$X=X(a,b)=3a^5-65a^4b-138a^3b^2+910a^2b^3-735ab^4-637b^5$$
$$Y=Y(a,b)=a^5-75a^4b-70a^3b^2+1050a^2b^3+245ab^4-735b^5$$

A few interesting properties observed are as follows:

1. $x(a(a+1),1)-y(a(a+1),1)=120f_{5,4}^{a(a+1)}-N_4(a(a+1))-3CI(a(a+1))-48CP_{a(a+1)}^{20}-180t_{3,a(a+1)}-286P(a)+102.$
2. $X(a(a+1),1)-x(a(a+1),1)-y(a(a+1),1)=24f_{5,7}^{a(a+1)}-24f_{4,3}^{a(a+1)}-2CP_{a(a+1)}^{9}-10cp_{27,a(a+1)}-1085P(a)+255.$
3. $X(a(a+1),1)-Y(a(a+1),1)-120f_{5,4}^{a(a+1)}-N_4(a(a+1))+3CD(a(a+1))-48P_{a(a+1)}^{13}+270t_{3,a(a+1)-1}+1572t_{3,a}$ is a perfect square.

Remark 1

In addition to (5), we have other choices of 8 written as the product of complex conjugates.

Choice 1

$8=\frac{(5+i\sqrt{7})(5-i\sqrt{7})}{4}$.The integer solutions to (1) are found to be

$$x=3a^5-5a^4b-210a^3b^2+70a^2b^3+735ab^4-49b^5$$
$$y=2a^5-30a^4b-140a^3b^2+420a^2b^3+490ab^4-294b^5$$
$$z=a^2+7b^2$$
$$X=\frac{1}{2}[11a^5-45a^4b-770a^3b^2+630a^2b^3+2695ab^4-441b^5]$$
$$Y=\frac{1}{2}[9a^5-95a^4b-630a^3b^2+1330a^2b^3+2205ab^4-931b^5]$$

Choice 2

$8=\frac{(11+i\sqrt{7})(11-i\sqrt{7})}{16}$.The non-zero distinct integer solutions to (1) are presented as follows:

$$x=3a^5+5a^4b-210a^3b^2-70a^2b^3+735ab^4+49b^5$$

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$$y = \frac{1}{2}[5a^5 - 45a^4b - 350a^3b^2 + 630a^2b^3 + 1225ab^4 - 441b^5$$

$z = a^2 + 7b^2$

$$X = \frac{1}{4}[23a^5 - 15a^4b - 1610a^3b^2 + 210a^2b^3 + 5635ab^4 - 147b^5]$$

$$Y = \frac{1}{4}[21a^5 - 125a^4b - 1470a^3b^2 + 1750a^2b^3 + 5145ab^4 - 1225b^5]$$

Choice 3

$8 = \frac{(31+i\sqrt{7})(31-i\sqrt{7})}{121}$.The integer solutions satisfied by (1) are given by

$$x = 468512A^5 + 1756920A^4B - 32795840A^3B^2 - 24596880A^2B^3 + 114785440AB^4 + 17217816B^5$$

$$y = 439230A^5 - 2781790A^4B - 30746100A^3B^2 + 38945060A^2B^3 + 107611350AB^4 - 27261542B^5$$

$$z = 121A^2 + 847B^2$$

$$X = 922383A^5 + 1244485A^4B - 64566810A^3B^2 - 17422790A^2B^3 + 225983835AB^4 + 12195953B^5$$

$$Y = 893101A^5 - 3294225A^4B - 62517070A^3B^2 + 46119150A^2B^3 + 218809745AB^4 - 32283405B^5$$

Pattern-II

Rewrite (3) as $u^2 + 7v^2 = 8z^5 * 1$ (6)

Write 1 as $1 = \frac{1}{16}(3 + i\sqrt{7})(3 - i\sqrt{7})$ (7)

Using (5) and (7) in (6) and following the procedure similar to Pattern-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$x = x(a,b) == -40a^4b + 560a^2b^3 - 392b^5$
$y = y(a,b) = -2a^5 - 30a^4b + 140a^3b^2 + 420a^2b^3 - 490ab^4 - 294b^5$
$z = z(a,b) = a^2 + 7b^2$
$X = X(a,b) = -a^5 - 75a^4b + 70a^3b^2 + 1050a^2b^3 - 245ab^4 - 735b^5$
$Y = Y(a,b) = -3a^5 - 65a^4b + 210a^3b^2 + 910a^2b^3 - 735ab^4 - 637b^5$

Properties

1. $y(a,1) - Y(a,1) = x(a,1) - X(a,1)$.
2. $y(a(a+1),1) - Y(a(a+1),1) = 120f_{5,3}^{a(a+1)} + 5N_4(a(a+1)) - 62PCS_{5,a(a+1)} + 30t_{31,a(a+1)} + 546t_{3,a} + 338$.
3. $z(a(a+1),1) - x(a(a+1),1)8 - N_4(a(a+1)) + 5TOH(a(a+1)) + 80t_{4,a(a+1)} + 80P(a)$is a perfect square.

Remark 2

Instead of (7), one may consider 1 as $1 = \frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{64}$ (8)

Following the procedure presented above, the integer solutions to (1) are obtained as

$$x = -2a^5 - 30a^4b + 140a^3b^2 + 420a^2b^3 - 490ab^4 - 294b^5$$

$y = -3a^5 - 5a^4b + 210a^3b^2 + 70a^2b^3 - 735ab^4 - 49b^5$
$z = a^2 + 7b^2$
$Y = \frac{1}{2}[-9a^5 - 95a^4b + 630a^3b^2 + 1330a^2b^3 - 2205ab^4 - 931b^5]$

$$X = \frac{1}{2}[-11a^5 - 45a^4b + 770a^3b^2 + 630a^2b^3 - 2695ab^4 - 441b^5]$$

Remark 3

Employing the other choices of complex conjugates for 8 and using (7) and (8) in turn, the other sets of non-zero distinct integer solutions to (1) may be obtained.

CONCLUSION

In linear transformations (2), the variables *X* and *Y* may also be represented by

$$X = 2uv + 1 \quad ; \quad Y = 2uv - 1$$

Applying the procedure similar to that of patterns I and II, other choices of integral solutions to (1) are obtained. To conclude, one may search for other patterns of solutions and their corresponding properties to heptic equations, homogeneous or non-homogeneous with variables ≥ 5.

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