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Research Article

ON A FAMILY OF D(1) STRONG RATIONAL DIOPHANTINE QUADRUPLES

Vidhyalakshmi, A. Kavitha and *M.A. Gopalan
**Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002*
**Author for Correspondence*

ABSTRACT
This paper concerns with the study of constructing a family of strong rational Diophantine quadruples (a,b,c,d) such that the product of any two elements of the set added with one is a perfect square.

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INTRODUCTION

Let q be a non-zero rational number. A set $\{a_1, a_1, \dots, a_m\}$ of non-zero rationals is called a rational $D(q)$ – m-tuple, if $a_i a_j + q$ is a square of a rational number for all $1 \le i \le j \le m$. The Greek mathematician Diophantus of Alexandria considered a variety of problems on indeterminant equations with rational or integer solutions. In particular, one of the problems was to find the sets of distinct positive rational numbers such that the product of any two numbers is one less than a rational square (Heath, 1964) and Diophantus found four positive rationals $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ (Dickson,1966; Bashmakova,1974). The first set of four positive integers with the same property, the set $\{1,3,8,120\}$ was found by Fermat. It was proved in 1969 by Baker and Davenport (Baker and Davenport,1969) that a fifth positive integer cannot be added to this set and one may refer (Dujella,1997; Dujella and Petho, 1998; Fujita, 2008) for generalization. However, Euler discovered that a fifth rational number can be added to give the following rational Diophantine quintuple $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$.Rational sextuples with two equal elements have been given in (Arkin *et al.*, 1993). Several examples of rational Diophantine sextupes, eg., $\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}$, $\left\{\frac{17}{448}, \frac{265}{448}, \frac{2145}{448}, 252, \frac{23460}{7}, \frac{2352}{7921}\right\}$ are found (Gibbs, 1978).
All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples (Arkin and Hoggatt, 1979; Gibbs, 1978) (this is known to be true for polynomials with integer co-efficients (Dujella and Fuchs, 2004). If so, then there are no Diophantine quintuples. However there are infinitely many irregular rational Diophantine quadruples. The smallest is $\frac{1}{4}, 5, \frac{33}{4}, \frac{105}{4}$. Many of these irregular quadruples are examples of another common type for which two of the sub-triples are regular i.e., $\{a,b,c,d\}$ is an irregular rational Diophantine quadruple, while $\{a,b,c,\}$ and $\{a,b,d\}$ are regular Diophantine triples. These are known as semi-regular rational Diophantine quadruples.There are only finitely many of these for any given common denominator l and they can be readily found.
Moreover in (Fujita, 2009) it has been proved that the $D(\mp k^2)$- triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$- quintuple. In (Flipin and Fujita, 2011) it has been proved that $D(-k^2)$- triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$- quadruple if $k \ge 5$. These results motivated us to search for strong rational Diophantine quadruples (Dujella and Vinko, 2008).

SECTION 1:

Let $a = \frac{2p-q}{q}, b = \frac{8(p-q)}{q}$ $p \ne q$ be two rational numbers such that ab+1 is a perfect square.
Let C_N be any non-zero rational number such that

$$\left(\frac{2p-q}{q}\right) C_N + 1 = \alpha_N^2 \tag{1}$$

$$8\left(\frac{p-q}{q}\right) C_N + 1 = \beta_N^2 \tag{2}$$

Eliminating C_N from (1) and (2), we obtain

Research Article

$$8\left(\frac{p-q}{q}\right)\alpha_N^2-\left(\frac{2p-q}{q}\right)\beta_N^2=\frac{6p-7q}{q} \tag{3}$$

Setting

$$\alpha_N=X_N+\left(\frac{2p-q}{q}\right)T_N \tag{4}$$

$$\beta_N=X_N+8\left(\frac{p-q}{q}\right)T_N \tag{5}$$

in (3), we get

$$X_N^2=\frac{8(p-q)(2p-q)}{q^2}T_N^2+1 \tag{6}$$

whose general solution is

$$\left.\begin{aligned} X_N&=\frac{1}{2}\left\{\left[\left(\frac{4p-3q}{q}\right)+\frac{\sqrt{8(p-q)(2p-q)}}{q}\right]^{N+1}+\left[\left(\frac{4p-3q}{q}\right)-\frac{\sqrt{8(P-q)(2p-q)}}{q}\right]^{N+1}\right\}\\ T_N&=\frac{q}{2\sqrt{8(p-q)(2p-q)}}\left\{\begin{matrix}\left[\left(\frac{4p-3q}{q}\right)+\frac{\sqrt{8(p-q)(2p-q)}}{q}\right]^{N+1}-\\ \left[\left(\frac{4p-3q}{q}\right)-\frac{\sqrt{8(P-q)(2p-q)}}{q}\right]^{N+1}\end{matrix}\right\}\end{aligned}\right\} \tag{7}$$

Substituting N=0 in (4) and (7) and employing (1) we get,

$$C_0=\frac{3(6p-5q)}{q}$$

Note that the triple (a,b,C_0) is a Diophantine triple with property D(1)
Again, substituting N=1,2 in (7) and (4) and employing (1) in turn we get,

$$C_1=\frac{8}{q^3}(12p^2-17pq+6q^2)(12p-11q)$$

$$C_2=\frac{1}{q^5}(192p^2-320pq+133q^2)(384p^3-832p^2q+586pq^2-135q^3)$$

It is seen that the quadruples (a,b,C_0,C_1) and (a,b,C_1,C_2) are strong rational Diophantine quadruples with property D(1). The repetition of the above process leads to the result that the quadruple(a,b,C_{N+1},C_N) , N=1,2,3….. is a family of strong rational Diophantine quadruple with property D(1). However, it is worth mentioning here that, for some particular values of p and q, the elements in the quadruple may be integers. A few examples are presented in the table below:

Table 1

(p, q)	(a,b,C_0,C_1)	(a,b,C_1,C_2)	(a,b,C_2,C_3)
(2,1)	(3,8,21,2080)	(3,8,2080,203841)	(3,8,203841,19974360)
(3,2)	(2,4,12,420)	(2,4,420,14280)	(2,4,14280,485112)
(5,3)	$\left(\frac{7}{3},\frac{16}{3},\frac{45}{3},792\right)$	$\left(\frac{7}{3},\frac{16}{3},792,\frac{9965025}{243}\right)$	$\left(\frac{7}{3},\frac{16}{3},\frac{9965025}{24},\frac{4642003080}{2187}\right)$
(8,5)	$\left(\frac{11}{5},\frac{24}{5},\frac{69}{5},\frac{78064}{125}\right)$	$\left(\frac{11}{5},\frac{24}{5},\frac{78064}{125},\frac{86339409}{3125}\right)$	$\left(\frac{11}{5},\frac{24}{5},\frac{86339409}{3125},\frac{95443690104}{78125}\right)$

SECTION 2:

Let $C_o=-\frac{2mn}{m^2+n^2}$, $C_1=\frac{2mn}{m^2+n^2}$, m>n>0 be two rational numbers such that
$C_0C_1+1=r^2$, is a perfect square. Let C_2 be any non-zero rational number such that

$$C_2 = C_0 + C_1 + 2r$$
$$C_2 = \frac{2(m^2-n^2)}{m^2+n^2}$$

Note that (C_0, C_1, C_2) is a strong rational Diophantine triple with property D(1)
Then, by Euler's solution the fourth tuple C_3 is given by

$$C_3 = 4r(r+C_0)(r+C_1)$$
$$C_3 = \frac{4(m^2-n^2)}{(m^2+n^2)^3}[(m^2-n^2)^2 - 4m^2n^2]$$

Now, $(C_0, C_1, C_2 C_3)$ forms a strong rational Diophantine quadruple with property D(1).
Consider, C_2 and C_3 such that $C_2C_3 + 1 = R^2$, a perfect square. If (C_2, C_3, C_4, C_5) is a strong rational Diophantine quadruple with property D(1), then it is seen that

$$C_4 = \frac{1}{(m^2+n^2)^3}[12m^6 - 40m^4n^2 + 12m^2n^4] \quad \text{and}$$
$$C_5 = \frac{(12m^4+12n^4-40m^2n^2)(5m^4-10m^2n^2+n^4)(7m^6+21m^2n^4-35m^4n^2-n^6)}{(m^2+n^2)^7}$$

Repeating this process again and again, one can generate a family of strong rational Diophantine quadruples with property D(1). A few numerical examples are presented in the table.

Table 2

(m, n)	(C_0, C_1, C_2, C_3)	(C_2, C_3, C_4, C_5)
(3,2)	$\left(\frac{-12}{13}, \frac{12}{13}, \frac{10}{13}, \frac{-2380}{2197}\right)$	$\left(\frac{10}{13}, \frac{-2380}{2197}, \frac{-2484}{2197}, \frac{55171572}{13^7}\right)$
(2,1)	$\left(\frac{-4}{5}, \frac{4}{5}, \frac{6}{5}, \frac{-84}{125}\right)$	$\left(\frac{6}{5}, \frac{-84}{125}, \frac{176}{125}, \frac{-52316}{78125}\right)$
(4,1)	$\left(\frac{-8}{17}, \frac{8}{17}, \frac{30}{17}, \frac{9660}{17^3}\right)$	$\left(\frac{30}{17}, \frac{9660}{17^3}, \frac{39104}{17^3}, \frac{54923247028}{17^7}\right)$

SECTION 3:

Let $C_o = -\frac{(m^2-n^2)}{m^2+n^2}$, $C_1 = \frac{(m^2-n^2)}{m^2+n^2}$, m>n>0 be two rational numbers such that
$C_0C_1 + 1 = r^2$, is a perfect square. Let C_2 be any non-zero rational number such that

$$C_2 = C_0 + C_1 + 2r = \frac{4mn}{(m^2+n^2)}$$

Following the analysis similar to Section.2, the corresponding strong rational Diophantine quadruples (C_0, C_1, C_2, C_3) and (C_2, C_3, C_4, C_5) with property D(1) are given by

$$C_3 = -\left(\frac{8m^5n - 48m^3n^3 + 8mn^5}{(m^2+n^2)^3}\right)$$
$$C_4 = \frac{2m^6 - 4m^5n - 26m^4n^2 + 56m^3n^3 - 26m^2n^4 - 4mn^5 + 2n^6}{(m^2+n^2)^3}$$
$$C_5 = \frac{1}{(m^2+n^2)^7}[4m^4 - 56m^2n^2 + 4n^4][m^4 - 14m^2n^2 + 4m^3n + 4mn^3 + n^4]$$

$$[m^6 - 8m^5n - 13m^4n^2 + 48m^3n^3 - 13m^2n^4 - 8mn^5 + n^6]$$

A few numerical examples are presented below:

Table 3

(m, n)	(C_0, C_1, C_2, C_3)	(C_2, C_3, C_4, C_5)
(2,1)	$\left(\frac{-3}{5}, \frac{3}{5}, \frac{8}{5}, \frac{112}{125}\right)$	$\left(\frac{8}{5}, \frac{112}{125}, \frac{-78}{125}, \frac{12948}{78125}\right)$
(4,2)	$\left(\frac{-12}{20}, \frac{12}{20}, \frac{32}{20}, \frac{7168}{20^3}\right)$	$\left(\frac{32}{20}, \frac{7168}{20^3}, \frac{-4992}{20^3}, \frac{212140032}{20^7}\right)$
(3,2)	$\left(\frac{-5}{13}, \frac{5}{13}, \frac{24}{13}, \frac{5712}{13^3}\right)$	$\left(\frac{24}{13}, \frac{5712}{13^3}, \frac{-814}{13^3}, \frac{65111860}{13^7}\right)$

Note: In sections.2 and 3, the repetition of the process leads to the generation of strong D(1) rational Diophantine quadruple $(C_{2m}, C_{2m+1}, C_{2m+2}, C_{2m+3})$, where m = 0, 1,2………..

Conclusion

In this paper, we have presented a family of strong rational Diophantine quadruples. To conclude, one may search for other families of strong and almost strong rational Diophantine quadruples with suitable property.

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