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Research Article

SOME NON-EXTENDABLE DIOPHANTINE TRIPLES IN SPECIAL NUMBER PATTERNS

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ABSTRACT
In this paper, we present three non-extendable Diophantine triples whose members are special numbers, namely, Triangular number, Jacobsthal number, Jacobsthal-Lucas number, Kynea number and Star number with suitable property.

Keywords: *Diophantine Triples, Integer Sequences*
2010 Mathematical subject classification; 11D99

Notations

$t_{m,n}$ - Polygonal number of rank n with size m.

ky_n -Kynea number of rank n

S_n -Star number of rank n

J_n -Jacobsthal number of rank n

j_n -Jacobsthal-Lucas number of rank n

INTRODUCTION

A Set of positive integers ($a_1, a_2, a_3, ...a_m$) is said to have the property D(n), $n \in z - \{0\}$, if $a_i a_j + n$ is a perfect square for all $1 \le i \lhd j \le m$ and such a set is called a Diophantine –m-tuple with property D(n). Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n[1] and also for any linear polynomial in n.In this context, one may refer (Thamotherampillai, 1980; Brown, 1985; Gupta and Singh, 1985; Beardon and Deshpande, 2002; Deshpande, 2002; Deshpande, 2003; Bugeaud *et al.*, 2007; Liqun, 2007; Fujita, 2008; Filipin *et al.*, 2012; Gopalan and Pandichelvi, 2011; Fujita and Togbe, 2011; Gopalan and Srividhya, 2012; Gopalan and Srividhya, 2012; Filipin, 2005; Gopalan *et al.*, 2014; Gopalan *et al.*, 2014; Pandichelvi, 2011) for an extensive review of various problem on Diophantine triples.

These results motivated us to search for non-extendable Diophantine triples with elements represented by special numbers, namely, polygonal numbers and other special number patterns.

In this paper, we exhibit three non-extendable Diophantine triples whose members are Triangular number, Jacobsthal number, Jacobsthal-Lucas number, Kynea number and Star number with suitable property.

METHOD OF ANALYSIS

I: Non-extendable $D(4t_{3,2^n} + 2^{2n+1} + 1)$ *Diophantine triple:*

Let $a = ky_n = (2^n + 1)^2 - 2$ and $b = j_{2n} - 1 = 2^{2n}$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(2^{2n})(c) + (4t_{3,2^n} + 2^{2n+1} + 1) = \alpha^2 \quad (1)$$

$$((2^n + 1)^2 - 2)(c) + (4t_{3,2^n} + 2^{2n+1} + 1) = \beta^2 \quad (2)$$

Eliminating 'c' from (1) and (2), we obtain

$$((2^n+1)^2-2)(\alpha^2)-(2^{2n})(\beta^2)=[((2^n+1)^2-2)-2^{2n}](4t_{3,2^n}+2^{2n+1}+1) \quad (3)$$

Using the linear transformations

$$\alpha = X+(2^{2n})T \quad (4)$$

$$\beta = X+((2^n+1)^2-2)T \quad (5)$$

in (3), it leads to the pell equation

$$X^2=((2^n+1)^2-2)(2^{2n})T^2+(4t_{3,2^n}+2^{2n+1}+1) \quad (6)$$

Let $T_0=1$ and $X_0=(2^n(2^n+1)+1)$ be the initial solution of (6). Thus (4) yields $\alpha_0=2(2^{2n})+2^n+1$

And using (1), we get $c=8t_{3,2^n}+1$

Hence (a, b, c)=($ky_n, j_{2n}-1, 8t_{3,2^n}+1$) is the Diophantine triple with property $D(4t_{3,2^n}+2^{2n+1}+1)$

Some numerical examples are presented below

n	(a, b, c) with property $D(4t_{3,2^n}+2^{2n+1}+1)$
1	(7, 4, 25) with property D(21)
2	(23, 16, 81)with property D(73)
3	(79, 64, 289)with property D(273)
4	(287, 256, 1089)with property D(1057)
5	(1087, 1024, 4225)with property D(4161)

We show that the above triple cannot be extended to quadruple
Let 'd' be any non-zero integer such that

$$((2^n+1)^2-2)(d)+(4t_{3,2^n}+2^{2n+1}+1)=p^2 \quad (7)$$

$$(2^{2n})(d)+(4t_{3,2^n}+2^{2n+1}+1)=q^2 \quad (8)$$

$$(4(2^{2n})+4(2^n)+1)(d)+(4t_{3,2^n}+2^{2n+1}+1)=r^2 \quad (9)$$

Eliminating 'd' from (8) and (9), we obtain

$$(4(2^{2n})+4(2^n)+1)(q^2)-(2^{2n})(r^2)=[4(2^{2n})+4(2^n)+1)-(2^{2n})]((4t_{3,2^n}+2^{2n+1}+1)) \quad (10)$$

Using the linear transformations

$$q=X+(4(2^{2n})+4(2^n)+1)T \quad (11)$$

$$r=X+(2^{2n})T \quad (12)$$

in (10), it leads to the pell equation

$$X^2=(4(2^{2n})+4(2^n)+1)(2^{2n})T^2+(4t_{3,2^n}+2^{2n+1}+1) \quad (13)$$

Let $T_0 = 1$ and $X_0 = 2(2^{2n}) + (2^n) + 1$ be the initial solution of (13). Thus (11) yields $q_0 = 3(2^{2n}) + 2^n + 1$

And using (8), we get $d = 9(2^{2n}) + 6(2^n + 1) - 3$

Verify Quadruple:

Substituting the above value of 'd' in L.H.S of (7), we have

$$ad + (4t_{3,2^n} + 2^{2n+1} + 1) = [3(2^{2n}) + 4(2^n)]^2 - (2^n - 1)^2 - (5(2^{2n}) + 1)$$

Note that the R.H.S is not a perfect square

Hence the triple $(ky_n, j_{2n} - 1, 8t_{3,2^n} + 1)$ with property $D(4t_{3,2^n} + 2^{2n+1} + 1)$ cannot be extended to a quadruple.

Note:

The triple $(ky_n, j_{2n} - 1, 8t_{3,2^n} + 1)$ is a strong Diophantine triple and the quadruple $(ky_n, j_{2n} - 1, 8t_{3,2^n} + 1, 9(j_{2n} - 1) + 3(2(2^n) + 1))$ is almost strong Diophantine quadruple.

II: Non-extendable $D(2^{2n})$ *Diophantine triple:*

Let $a = 8t_{3,n} = 4n^2 + 4n$ and $b = j_{2n} - 1 = 2^{2n}$ be two integers such that ab+1 is a perfect square

Let 'c' be any non-zero integer such that

$$(2^{2n})c + 2^{2n} = \alpha^2 \tag{14}$$

$$(4n^2 + 4n)(c) + 2^{2n} = \beta^2 \tag{15}$$

Eliminating 'c' from (14) and (15), we obtain

$$(4n^2 + 4n)(\alpha^2) - (2^{2n})(\beta^2) = [((4n^2 + 4n) - 2^{2n})2^{2n}] \tag{16}$$

Using the linear transformations

$$\alpha = X + (2^{2n})T \tag{17}$$

$$\beta = X + (4n^2 + 4n)T \tag{18}$$

in (16), it leads to the pell equation

$$X^2 = (4n^2 + 4n)(2^{2n})T^2 + (2^{2n}) \tag{19}$$

Let $T_0 = 1$ and $X_0 = (2n + 1)(2^n)$ be the initial solution of (19). Thus (17) yields $\alpha_0 = (2^n)(2n + 1) + 2^{2n}$

And using (14), we get $c = 8t_{3,n} + (j_{2n} - 1) + (2n + 1)(3J_n + j_n)$

Hence (a, b, c)=($8t_{3,n}, j_{2n} - 1, 8t_{3,n} + (j_{2n} - 1) + (2n + 1)(3J_n + j_n)$) is the Diophantine triple with property $D(2^{2n})$

Some numerical examples are presented below

n	(a, b, c) with property $D(2^{2n})$
1	(8, 4, 24) with property D(4)
2	(24, 16, 80) with property D(16)
3	(48, 64, 224) with property D(64)
4	(80, 256, 624) with property D(256)
5	(120, 1024, 1848) with property D(1024)

We show that the above triple cannot be extended to quadruple
Let ‘d’ be any non-zero integer such that

$$(4n^2+4n)(d)+2^{2n}=p^2 \quad (20)$$

$$(2^{2n})(d)+2^{2n}=q^2 \quad (21)$$

$$(4n^2+4n+(2^{2n})+(4n+2)2^n)(d)+2^{2n}=r^2 \quad (22)$$

Eliminating ‘d’ from (21) and (22), we obtain

$$(4n^2+4n+2^{2n}+(4n+2)2^n)(q^2)-(2^{2n})(r^2)=[4n^2+4n+(4n+2)2^n](2^{2n}) \quad (23)$$

Using the linear transformations

$$q=X+(2^{2n})T \quad (24)$$

$$r=X+(4n^2+4n+2^{2n}+(4n+2)2^n)T \quad (25)$$

in (23), it leads to the pell equation

$$X^2=(4n^2+4n+2^{2n}+(4n+2)2^n)2^{2n}T^2+2^{2n} \quad (26)$$

Let $T_0=1$ and $X_0=2^n(2n+1)+(2^{2n})$ be the initial solution of (26). Thus (24) yields $q_0=2(2^{2n})+2^n(2n+1)$

And using (21), we get $d=8t_{3,n}+4[(j_{2n}-1)+(2n+1)(3J_n+j_n)]$

Verify Quadruple:

Substituting the above value of ‘d’ in L.H.S of (20), we have

$$ad+2^{2n}=[4n^2+4n(2^n+1)+2^n]^2+8n2^{2n}+8n2^n+8n^2 2^n$$

Note that the R.H.S is not a perfect square

Hence the triple ($8t_{3,n}, j_{2n}-1, 8t_{3,n}+(j_{2n}-1)+(2n+1)(3J_n+j_n)$) with property $D(2^{2n})$ cannot be extended to a quadruple.

Note:

The triple ($8t_{3,n}, j_{2n}-1, 8t_{3,n}+(j_{2n}-1)+(2n+1)(3J_n+j_n)$) is a strong Diophantine triple and the quadruple

$(8t_{3,n}, j_{2n}-1, 8t_{3,n}+(j_{2n}-1)+(2n+1)(3J_n+j_n), 8t_{3,n}+4[(j_{2n}-1)+(2n+1)(3J_n+j_n)])$ is almost strong Diophantine quadruple.

II: Non-extendable $D(3n^2)(2^{2n})$ *Diophantine triple:*

Let $a = S_n = 6n^2 + 6n - 1$ and $b = j_{2n} - 1 = 2^{2n}$ be two integers such that ab+1 is a perfect square
Let 'c' be any non-zero integer such that

$$(2^{2n})c + (3n^2)(2^{2n}) = \alpha^2 \tag{27}$$

$$(6n^2 - 6n + 1)(c) + (3n^2)(2^{2n}) = \beta^2 \tag{28}$$

Eliminating 'c' from (27) and (28), we obtain

$$(6n^2 - 6n + 1)(\alpha^2) - (2^{2n})(\beta^2) = [((6n^2 - 6n + 1) - 2^{2n})(3n^2)(2^{2n})] \tag{29}$$

Using the linear transformations

$$\alpha = X + (2^{2n})T \tag{30}$$

$$\beta = X + (6n^2 - 6n + 1)T \tag{31}$$

in (29), it leads to the pell equation

$$X^2 = (6n^2 - 6n + 1)(2^{2n})T^2 + (3n^2)(2^{2n}) \tag{32}$$

Let $T_0 = 1$ and $X_0 = (3n - 1)(2^n)$ be the initial solution of (32). Thus (30) yields $\alpha_0 = (2^n)(3n - 1) + 2^{2n}$

And using (27), we get $c = S_n + (j_{2n} - 1) + (3n - 1)(3J_n + j_n)$

Hence (a, b, c)=($S_n, j_{2n} - 1, S_n + (j_{2n} - 1) + (3n - 1)(3J_n + j_n)$) is the Diophantine triple with property $D(3n^2)(2^{2n})$

Some numerical examples are presented below

n	(a, b, c) with property $D(3n^2)(2^{2n})$
1	(1, 4, 13) with property D(12)
2	(13, 16, 69) with property D(192)
3	(37, 64, 229) with property D(1728)
4	(73, 256, 681) with property D(12288)
5	(121, 1024, 2041) with property D(76800)

We show that the above triple cannot be extended to quadruple
Let 'd' be any non-zero integer such that

$$(6n^2 - 6n + 1)(d) + (3n^2)(2^{2n}) = p^2 \tag{33}$$

$$(2^{2n})(d) + (3n^2)(2^{2n}) = q^2 \tag{34}$$

$$(6n^2 - 6n + 1 + (6n - 2)(2^n) + (2^{2n}))(d) + (3n^2)(2^{2n}) = r^2 \tag{35}$$

Eliminating 'd' from (34) and (35), we obtain

$$(6n^2 - 6n + 1 + (6n - 2)(2^n) + (2^{2n}))(q^2) - (2^{2n})(r^2) = [(6n^2 - 6n + 1 + (6n - 2)(2^n))](3n^2)(2^{2n}) \tag{36}$$

Using the linear transformations

$$q = X + (2^{2n})T \tag{37}$$

$$r = X + (6n^2 - 6n + 1 + (6n - 2)(2^n) + (2^{2n}))T \tag{38}$$

in (36), it leads to the pell equation

$$X^2 = (2^{2n})(6n^2 - 6n + 1 + (6n-2)(2^n) + (2^{2n}))T^2 + (3n^2)(2^{2n}) \tag{39}$$

Let $T_0 = 1$ and $X_0 = 2^n(3n-1) + (2^{2n})$ be the initial solution of (39). Thus (37) yields $q_0 = 2(2^{2n}) + 2^n(3n-1)$

And using (34), we get $d = S_n + 4(j_{2n} - 1) + (3n-1)(3J_n + j_n)]$

Verify Quadruple:
Substituting the above value of 'd' in L.H.S of (33), we have

$$ad + (3n^2)(2^{2n}) = [(S_n + (3n-1)(3J_n + j_n)]^2 - 9n^2 2^{2n}$$

Note that the R.H.S is not a perfect square

Hence the triple ($S_n, j_{2n} - 1, S_n + (j_{2n} - 1) + (3n-1)(3J_n + j_n)$) with property $D(3n^2)(2^{2n})$ cannot be extended to a quadruple.

Note:
The triple $(S_n, j_{2n} - 1, S_n + (j_{2n} - 1) + (3n-1)(3J_n + j_n))$ is a strong Diophantine triple and the quadruple $\{S_n, j_{2n} - 1, S_n + (j_{2n} - 1) + (3n-1)(3J_n + j_n), S_n + 4(j_{2n} - 1) + (3n-1)(3J_n + j_n)\})$ is almost strong Diophantine quadruple.

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