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***Research Article***

# CUBIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS

$$(X^3+Y^3)-(Z^3+W^3)=39(X+Y)T^2$$

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## ABSTRACT
The cubic equation $(X^3+Y^3)-(Z^3+W^3)=39(X+Y)T^2$ is analyzed for its patterns of non-zero integer solutions. Four patterns of solutions are illustrated. A few properties among the solutions are presented.

***Keywords:*** *Cubic Equation with Five Unknowns, Integral solutions*

**MSC 2010 subject classification: 11D25**

## INTRODUCTION
Integral solutions for the homogeneous or non-homogeneous Diophantine cubic equation is an interesting concept as it can seen from (Dickson, 1952; Carmichael, 1959; Mordell, 1969). In (Gopalan and Premalatha, 2009; Gopalan and Pandichelvi, 2010; Gopalan and Sivagami, 2010; Gopalan and Premalatha, 2010; Gopalan and Kalingarani, 2010; Gopalan and Premalatha, 2010; Gopalan *et al.*, 2012) a few special cases of cubic Diophantine equations with 4 unknowns are studied. In (Gopalan *et al.*, 2012; Gopalan *et al.*, 2013; Gopalan *et al.*, 2014) cubic equations cubic with 5 unknowns are studied for their integral solutions.
This communication concerns with yet another cubic Diophantine equation with five unknowns $(X^3+Y^3)-(W^3+Z^3)=39(X+Y)T^2$

## NOTATIONS:
$T_{m,n}$ - Polygonal number of rank n with size m
$Pr_n$ - Pronic number of rank n
$Ct_n$ - Centered square number

## METHOD OF ANALYSIS
The cubic Diophantine equation with five unknowns to be solved is given by

$$(X^3+Y^3)-(W^3+Z^3)=39(X+Y)T^2 \qquad (1)$$

The substitution of the linear transformations

$$X=u+v,\ Y=u-v,\ Z=u+p,\ W=u-p \qquad (2)$$

in (1) leads to

$$v^2=p^2+13T^2 \qquad (3)$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.
***Pattern I:***
Equation (3) can be written as

$$v^2 - p^2 = 13T^2 \tag{4}$$

Factorizing (4), we get

$$(v+p)(v-p) = 13T.T \tag{5}$$

which is equivalent to the system of double equations,

$$\left.\begin{aligned} vB + pB - 13TA = 0 \\ TB - vA + pA = 0 \end{aligned}\right\} \tag{6}$$

Applying the method of cross multiplication, we get

$$v = -13A^2 + AB$$

$$p = -14AB$$

$$T = -B^2 - AB$$

Substituting the above values in (2), the corresponding solutions are,

$$X = X(u, A, B) = u - 13A^2 + AB$$

$$Y = Y(u, A, B) = u + 13A^2 - AB$$

$$Z = Z(u, A, B) = u - 14AB$$

$$W = W(u, A, B) = u + 14AB$$

$$T = T(A, B) = -B^2 - AB$$

***Pattern II:***

Instead of (3), write as

$$v^2 - p^2 = 13T^2 \tag{7}$$

Factorizing (7), we get

$$(v+p)(v-p) = 13T.T \tag{8}$$

which is equivalent to the system of double equations,

$$\left.\begin{aligned} vB + pB - TA = 0 \\ 13TB - vA + pA = 0 \end{aligned}\right\} \tag{9}$$

Applying the method of cross multiplication, we get

$$v = -A^2 + AB$$

$$p = -14AB$$

$$T = -13B^2 - AB$$

Substituting the above values in (2), the corresponding solutions are,

$$X = X(u, A, B) = u - A^2 + AB$$

$$Y = Y(u, A, B) = u + A^2 - AB$$

$$Z = Z(u, A, B) = u - 14AB$$

$$W = W(u, A, B) = u + 14AB$$

$$T = T(A, B) = -13B^2 - AB$$

***Pattern III:***

Equation (3) can be written as

$$v^2 - 13T^2 = p^2 * 1 \tag{10}$$

Assume $p = a^2 - 13b^2$ (11)

Write 1 as

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$$1=\frac{(7+\sqrt{13})(7-\sqrt{13})}{36} \quad (12)$$

Substituting (11) and a (12) in (10) and employing the method of factorization, define

$$v+\sqrt{13}T=(a+\sqrt{13}b)^2\frac{(7+\sqrt{13})}{6}$$

Equating rational and irrational factors, we get

$$\left.\begin{aligned} v&=\frac{1}{6}[7a^2+91b^2+26ab] \\ T&=\frac{1}{6}[a^2+13b^2+14ab \end{aligned}\right\} \quad (13)$$

As our interest is on finding integer solutions, we choose A and B suitably so that the values of v, T and p are in integers.

Replacing A by 6A and B by 6B in (11) and (13), the corresponding integer solutions of (3) in two parameters are

$$X=X(u,A,B)=u+42A^2+546B^2+156AB$$

$$Y=Y(u,A,B)=u-42A^2-546B^2-156AB$$

$$Z=Z(u,A,B)=u+A^2-13B^2$$

$$W=W(u,A,B)=u-A^2+13B^2$$

$$T=T(A,B)=6A^2+78B^2+84AB$$

***Properties:***

- $X(u,1,B)-Z(u,1,B)-84T_{4,A}\equiv 0(Mod 2)$
- $Z(u,A,1)-W(u,A,1)-Ct_{4,A}\equiv -1(Mod 2)$
- $T(A,1)-T_{14,A}\equiv 11A(Mod 39)$
- $Z(u,1,B)-Y(u,1,B)-533\Pr_B\equiv 2B(Mod 3)$
- $Z(u,1,B)-W(u,1,B)+26T_{4,B}\equiv 0(Mod 2)$

***Pattern IV:***

Instead of (3), write as

$$p^2+13T^2=v^2*1 \quad (14)$$

Assume $p=a^2+13b^2$ (15)

Write 1 as

$$1=\frac{(6+i\sqrt{13})(6-i\sqrt{13})}{49} \quad (16)$$

Substituting (15) and (16) in (14) and employing the method of factorization, define

$$p+\sqrt{13}T=(a+i\sqrt{13}b)^2\frac{(6+i\sqrt{13})}{7}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} p &= \frac{1}{7}[6a^2 - 78b^2 - 26ab] \\ T &= \frac{1}{7}[a^2 - 13b^2 + 12ab \end{aligned}\right\} \quad (17)$$

As our interest is on finding integer solutions, we choose A and B suitably so that the values of v, T and p are in integers.

Replacing A by 7A and B by 7B in (15) and (17), the corresponding integer solutions of (3) in two parameters are

$$X = X(u, A, B) = u + A^2 + 13B^2$$
$$Y = Y(u, A, B) = u - A^2 - 13B^2$$
$$Z = Z(u, A, B) = u + 42A^2 - 546B^2 - 182AB$$
$$W = W(u, A, B) = u - 7A^2 + 91B^2 - 84AB$$
$$T = T(A, B) = 7A^2 - 91B^2 + 84AB$$

***Properties***:

- $X(u,1,B) - Y(u,1,B) - T_{54,B} \equiv 2(Mod 25)$
- $X(u, B+1, B) - W(u, B+1, B) + 35T_{6,B} - 168T_{3,B} \equiv 8(Mod 19)$
- $Z(u, A, A+1) - W(u, A, A+1) + 1372T_{3,A} \equiv 1(Mod 2)$
- $T(A,1) - T_{16,A} \equiv -1(Mod 3)$
- $Z(u, A,1) - T_{86,A} \equiv 0(Mod 3)$

## CONCLUSION

In this paper, we have presented four patterns of the non-zero distinct integer solutions of the homogeneous Diophantine cubic equation given by $(X^3 + Y^3) - (Z^3 + W^3) = 39(X + Y)T^2$. To conclude, one may search for other patterns of solution and their corresponding properties.

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