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Research Article

OBSERVATIONS ON THE BIQUADRATIC EQUATION WITH FIVE UNKNOWNS

$$2(x^3+y^3)=(k^2+3s^2)(z^2-w^2)p^2$$

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ABSTRACT
We obtain infinitely many non-zero integer quintuples (x, y, z, w, p) satisfying the Biquadratic equation with five unknowns $2(x^3+y^3)=(k^2+3s^2)(z^2-w^2)p^2$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Star numbers, Stella Octangular numbers, Octahedral numbers, Four Dimensional Figurative numbers, Centered polygonal and pyramidal numbers are exhibited.

Key words: *Biquadratic equation with four unknowns, Integral solutions, polygonal and pyramidal numbers, Four Dimensional Figurative numbers, Centered polygonal and pyramidal numbers*

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NOTATIONS:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

GN_n - Gnonomic number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

PR_n - Pronic number of rank n .

$F_{4,n,3}$ - Four Dimensional Figurative number of rank n whose generating polygon is a triangle

$F_{4,n,4}$ - Four Dimensional Figurative number of rank n whose generating polygon is a square

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$CP_{n,7}$ - Centered heptogonal pyramidal number of rank n

$CP_{n,10}$ - Centered decagonal pyramidal number of rank n

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Carmichael 1959, Dickson 1952, Mordell 1959, Telang 1996) In this context one may refer Gopalan *et al.*, (2009, 2010 a,b,c, 2012) for various problems on the biquadratic diophantine equations with four variables. Gopalan and Kaligarani (2009) have discussed the problem on biquadratic Diophantine equation with five unknowns. This paper concerns with yet another problem of determining non-trivial integral solutions of the non-homogeneous biquadratic equation with five unknowns given by $2(x^3+y^3)=(k^2+3s^2)(z^2-w^2)p^2$.A few relations among the solutions are presented.

MATERIALS AND METHODS

Method of Analysis

The Diophantine equation representing the biquadratic equation under consideration with five unknowns is given by

$$2(x^3+y^3)=(k^2+3s^2)(z^2-w^2)p^2 \tag{1}$$

Introducing the linear transformations

$$x=u+v, y=u-v, z=u+1, w=u-1, \tag{2}$$

in (1) it simplifies to

$$u^2+3v^2=(k^2+3s^2)p^2 \tag{3}$$

The above equation (3) is solved through different approaches and thus, one obtains distinct sets of integer solutions to (1)

Case1: k^2+3s^2 ***Is Not A Perfect Square***

Approach1: Let $p=a^2+3b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(u+i\sqrt{3}v)=(k+i\sqrt{3}s)(a+i\sqrt{3}b)^2$$

Equating real and imaginary parts, we have

$$u=k(a^2-3b^2)-6sab$$

$$v=s(a^2-3b^2)+2kab$$

Substituting the values of u and v in (2), the non zero distinct integral solutions of (1) are given by

$$x=k(a^2-3b^2+2ab)+s(a^2-3b^2-6ab)$$

$$y=k(a^2-3b^2-2ab)-s(a^2-3b^2+6ab)$$

$$z=k(a^2-3b^2)-6sab+1 \tag{5}$$

$$w=k(a^2-3b^2)-6sab-1$$

$$p=a^2+3b^2$$

Approach2: (3) Can be written as

$$u^2+3v^2=(k^2+3s^2)p^2\times 1 \tag{6}$$

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Write 1 as

$$1=\frac{((1+i\sqrt{3})(1-i\sqrt{3})}{2^2} \tag{7}$$

Using (7) in (6) and employing the method of factorization, define

$$\left(u+i\sqrt{3}v\right)=\frac{(1+i\sqrt{3})}{2}(k+i\sqrt{3}s)(a+i\sqrt{3}b)^2 \tag{8}$$

Equating real and imaginary parts in (8) we get

$$\begin{aligned} u&=\frac{1}{2}[\{k(a^2-3b^2)-6sab\}-3\{s(a^2-3b^2)+2kab\}] \\ v&=\frac{1}{2}[\{k(a^2-3b^2)-6sab\}+\{s(a^2-3b^2)+2kab\}] \end{aligned} \tag{9}$$

Using (9) and (2), we get the integral solution of (1) as

$$\left.\begin{aligned} x&=4(k-s)(a^2-3b^2)-8(3s+k)ab \\ y&=-8s(a^2-3b^2)-16kab \\ z&=2(k-3s)(a^2-3b^2)-12(s+k)ab+1 \\ z&=2(k-3s)(a^2-3b^2)-12(s+k)ab+1 \\ p&=4(a^2+3b^2) \end{aligned}\right\} \tag{10}$$

The quintuple (x,y,z,w,p) *in (10) satisfies the following interesting properties:*

1. $x(a,a)+y(a,a)+z(a,a)+w(a,a)+64k(2P_a^5-CP_{a,6})=0$

2. The following expressions are nasty numbers:

(a) $3[x(a,a)-y(a,a)+z(a,a)-w(a,a)+32s(GN_a-2CP_{a,3}+CP_{a,6})]$

(b) $s[y(a^2,1)+16kT_{4.a}+8s(6F_{4,a,4}-6P_a^5+T_{4,a})]$

(c) $2[p(a(a+1),1)-24P_a^5+8T_{3,a})]$

3. The following expressions are cubic integer:

(a) $2[4(x+y)^2-3(z+w)^2-zw]$

(b) $4(x-2w+y)$

4. $8[2(x+y)+w^2-zw-z-w]$ is a biquadratic integer.

5. $p(a,1)+z(a,1)-w(a,1)-6P_a^5+CP_{a,6}+4T_{3,a}\equiv 0(\bmod 5)$

6. $x(a^2,1)-4(k-s)[F_{4,a,3}-18P_a^4-4T_{3.a}-6T_{4,a}+3T_{6,a}]+8(3k+s)(2P_a^5-CP_{a,6})\equiv 0$

7. $16\,j_{4n}-p(2^{2n},2^{2n})+z(2^{2n},2^{2n})-w(2^{2n},2^{2n})\equiv 0(\bmod 14)$

8. $x(2^{2n},2^{2n})+y(2^{2n},2^{2n})-4(3s-4k)(KY_{2n}-3J_{4n+1})\equiv 0(\bmod 8)$

Approach3: Instead of (7), write 1 as

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{7^2}$$

Following the similar procedure as in approach2, the corresponding integer solution of (1) are as follows:

$$\left.\begin{aligned} x &= 7[(5k-11s)(a^2-3b^2)-4ab(k-3s)4ab] \\ y &= -7[(3k+13s)(a^2-3b^2)] \\ z &= 7[(k-12s)(a^2-3b^2)+2ab(k-3s)]+1 \\ w &= 7[(k-12s)(a^2-3b^2)+2ab(k-3s)]-1 \\ p &= 7^2(a^2+3b^2) \end{aligned}\right\} \qquad (11)$$

Approach4: Suppose we write 1 as

$$1=\frac{(13+i3\sqrt{3})(13-i3\sqrt{3})}{14^2} \qquad (12)$$

Following the similar procedure as in approach2, the corresponding integer solution of (1) are as follows:

$$\left.\begin{aligned} x &= 14[2(8k+2s)(a^2-3b^2)+2ab(k-3s)4ab] \\ y &= 14[2(5k-11s)(a^2-3b^2)+2ab(k-3s)4ab] \\ z &= 14[(13k-9s)(a^2-3b^2)+2ab(k-3s)]+1 \\ w &= 14[(13k-9s)(a^2-3b^2)+2ab(k-3s)]-1 \\ p &= 14^2(a^2+3b^2) \end{aligned}\right\} \qquad (13)$$

Approach5: Suppose we write 1 as

$$1=\frac{(1+i15\sqrt{3})(1-i15\sqrt{3})}{26^2}$$

For this choice, the corresponding integer solutions are

$$\left.\begin{aligned} x &= 26[2(8k-22s)(a^2-3b^2)-8(11k+12s)ab] \\ y &= 26[-(14k+46s)(a^2-3b^2)+4(21s-23k)ab] \\ z &= 26[(k-45s)(a^2-3b^2)-2ab(45k+3s)]+1 \\ w &= 26[(k-45s)(a^2-3b^2)+2ab(45k+3s)]-1 \\ p &= 26^2(a^2+3b^2) \end{aligned}\right\} \qquad (14)$$

Approach6: Rewriting (3) we get,

$$u^2-(kp)^2=3\,[(sp)^2-v^2] \qquad (15)$$

Using the method of factorisation, writing (15) as a system of double equations, solving it by method of cross ratio and using (2), we get the solution of (1) as

$$\left.\begin{aligned} x &= (k-s)\alpha^2 + 3\beta^2(s-k) + 2\alpha\beta(3s+k) \\ y &= (k+s)\alpha^2 - 3\beta^2(s+k) + 2\alpha\beta(3s-k) \\ z &= k(\alpha^2 - 3\beta^2) + 6s\alpha\beta + 1 \\ w &= k(\alpha^2 - 3\beta^2) + 6s\alpha\beta - 1 \\ p &= \alpha^2 + 3\beta^2 \end{aligned}\right\} \qquad (16)$$

The quintuple (x, y, z, w, p) *in (16) satisfies the following interesting properties:*

1. The following expressions are nasty numbers:

(a) $12k[2kS_a + 24T_{3,a} - 12T_{4,a} + 72s(2CP_{a,6} - SO_a - 36k) - 6x(a,1) - 6y(a,1)]$

(b) $6(s-k)[x(a,a) - y(a,a) - 2(k-s)\{CT_{a,4} - 5CP_{a,6} + 3CP_{a,10})]$

(c) $s[x(a(a+1),1) - y(a(a+1),1) + 2s(24F_{4,a,3} - 24P_a^3 + 4T_{3,a}) - 4kPR_a]$

2. $x(a,a) + y(a,a) + z(a,a) + w(a,a) + 8(k+3s)[(3(OH_a) - 6P_a^4 + 4T_{4,a}] \equiv 0$

3. $\left.\begin{aligned} 216k^2[2k(24F_{4,a,3} - 3SO_a - 11GN_a + 2T_{4,a} - 2T_{6,a} \\ + 12s(2P_a^5 - CP_{a,6}) - x(a^2,1) - y(a^2,1)] \end{aligned}\right\}$ is a cubic integer.

4. $8[p(2^{2n}, 2^{2n}) + z(2^{2n}, 2^{2n}) - w(2^{2n}, 2^{2n}) - 4(KY_{2n} - j_{2n+1})]$ is a biquadratic integer

5. $z(a^2,1) - w((a^2,1) + p(a^2,1) + 6P_a^5 - 6F_{4,a,6} - T_{4,a} \equiv 0 \pmod 5)$

Approach7: Rewriting (3) as

$$u^2 - (kp)^2 = 3\,[(sp)^2 - v^2] \qquad (17)$$

Let $p = \alpha$. Using the method of factorisation, writing (15) as a system of double equations, solving it and using (2) we get the solution of (1) as

$$\left.\begin{aligned} x &= 4s\alpha \\ y &= 2\alpha(s+k) \\ z &= 3s\alpha + k\alpha + 1 \\ w &= 3s\alpha + k\alpha - 1 \\ p &= 2\alpha \end{aligned}\right\} \qquad (18)$$

Case2: $k^2 + 3s^2$ ***Is A Perfect Square***

Approach8: Choose k and s such that

$$k^2 + 3s^2 = d^2. \qquad (19)$$

Substituting (19) in (3) we get

$$u^2 + 3v^2 = (dp)^2 \qquad (20)$$

which is in the form of Pythagorean equation, whose solution is,

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$$u=3\alpha^2-\beta^2, v=2\alpha\beta, dp=3\alpha^2+\beta^2, \quad \alpha>\beta>0 \tag{21}$$

Taking $\alpha=d\alpha, \quad \beta=d\beta$ and using (2) we get the integral solutions (1) as

$$\left.\begin{aligned} x&=d^2(3\alpha^2-\beta^2+2\alpha\beta)\\ y&=d^2(3\alpha^2-\beta^2-2\alpha\beta)\\ z&=d^2(3\alpha^2-\beta^2)+1\\ w&=d^2(3\alpha^2-\beta^2)-1\\ p&=d(3\alpha^2+\beta^2)\end{aligned}\right\} \tag{22}$$

Approach9: Assuming

$$u=du, \quad v=dv \tag{23}$$

in (20), we get, $u^2+3v^2=p^2$ (24)

which is in the form of Pythagorean equation, whose solution is,

$$u=3\alpha^2-\beta^2, v=2\alpha\beta, p=3\alpha^2+\beta^2, \quad \alpha>\beta>0 \tag{25}$$

Using (23), (21) and (2) we get the integral solutions (1) as

$$\left.\begin{aligned} x&=d(3\alpha^2-\beta^2+2\alpha\beta)\\ y&=d(3\alpha^2-\beta^2-2\alpha\beta)\\ z&=d(3\alpha^2-\beta^2)+1\\ w&=d(3\alpha^2-\beta^2)-1\\ p&=3\alpha^2+\beta^2\end{aligned}\right\} \tag{26}$$

Approach10: Write (20) as

$$(dp)^2-3v^2=u^2 \tag{27}$$

Let $u=a^2-3b^2$ (28)

Substituting (27) in (28) and using the method of factorisation, define

$$(dp+\sqrt{3}v)=(a+\sqrt{3}b)^2 \tag{29}$$

Equating rational and irrational parts, we have

$$\left.\begin{aligned} dp&=a^2+3b^2\\ v&=2ab\end{aligned}\right\} \tag{30}$$

Taking $a=da, \quad \mathrm{b}=db$ and using (29), (28) and (2) we get the integral solutions (1) as

$$\left.\begin{aligned} x&=d^2(a^2-3b^2+2ab)\\ y&=d^2(a^2-3b^2-2ab)\\ z&=d^2(a^2-3b^2)+1\\ w&=d^2(a^2-3b^2)-1\\ p&=d(a^2+3b^2)\end{aligned}\right\} \tag{31}$$

Rewrite (29) as

$$(dp+\sqrt{3}v)=(a+\sqrt{3}b)^2\times 1$$

By using the procedure similar to that in approaches 2-5 we get 4 more patterns of solutions to (1).

Approach11: Applying the linear transformation

$$u = X + 3T,\ \ \mathrm{v} = X - T \tag{32}$$

In (20), we get,

$$4X^2 + 12T^2 = (dp)^2$$

Substituting $p = 2P$, the above equation reduces to Pythagoreans form, whose solution is

$$X = 3\alpha^2 - \beta^2, T = 2\alpha\beta, dP = 3\alpha^2 + \beta^2,\quad \alpha > \beta > 0 \tag{33}$$

Using (31), (32) & (2) and performing some algebra the corresponding integer solution is obtained as follows:

$$\left.\begin{aligned} x &= 2d^2(3\alpha^2 - \beta^2 + 2\alpha\beta) \\ y &= 8d^2\alpha\beta \\ z &= d^2(3\alpha^2 - \beta^2 + 6\alpha\beta) + 1 \\ w &= d^2(3\alpha^2 - \beta^2 + 6\alpha\beta) - 1 \\ p &= 2d(3\alpha^2 + \beta^2) \end{aligned}\right\} \tag{34}$$

Note

Write (20) as

$$(dp)^2 - u^2 = 3v^2,$$

which is written as the system of double equations

$$dp + u = 3v,\quad dp - u = v \tag{35}$$

Solving the above system (34) and using (2), the integral solution to (1) are obtained as
$(2d\alpha, 0, d\alpha + 1, d\alpha - 1, 2\alpha)$ and $(0, 2d\alpha, d\alpha + 1, d\alpha - 1, 2\alpha)$

Remark

If $(x_0, y_0, z_0, w_0, p_0)$ is a given solution to (1), then the quintuple $(x_1, y_1, z_1, w_1, p_1)$ also satisfies (1), where

$$x_1 = y_0 + (k^2 + 3s^2)p_0^2(z_0 + w_0)$$

$$y_1 = x_0 - (k^2 + 3s^2)p_0^2(z_0 + w_0)$$

$$z_1 = z_0 + 3(x_0 + y_0)[y_0 - x_0 + (k^2 + 3s^2)p_0^2(z_0 + w_0)]$$

$$w_1 = w_0 - 3(x_0 + y_0)[y_0 - x_0 + (k^2 + 3s^2)p_0^2(z_0 + w_0)]$$

$$p_1 = p_0$$

CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

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