COMPUTATION OF LIE DERIVATIVES IN NONLINEAR CONTROL SYSTEMS USING ALGORITHMIC DIFFERENTIATION

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ABSTRACT
The purpose of this study is to give the alternative method for calculating Lie derivatives in nonlinear control systems. Lie derivatives are often used in nonlinear control of mathematics and physics theories. The Lie derivatives are usually computed symbolically by computer algebra software. This method is able to solve the small and medium-size problems. However, the use of symbolic computation in nonlinear control systems due to the very complicated expressions in higher order derivatives is limited. In order to overcome these problems, an algorithmic differentiation is introduced to compute the mixed Lie derivatives in nonlinear control systems.

Keywords: Lie Derivatives, Algorithmic Differentiation, Nonlinear Control

INTRODUCTION
Lie derivatives are often used in nonlinear control and system theory of many methods in mathematics and physics equations (Isidori, 1995; Nijmeijer and van der Schaft, 1990; Röbenack, 2010). Lie derivatives can be computed symbolically by computer algebra packages such as Mathematica, Maple and MuPAD (Gómez, 1994; Kwatny and Blankenship, 2000). Symbolically computed solutions in small or/and medium problems can be carried out as source code of the programming languages C and Fortran (Röbenack, 2008). Although, the symbolic computation can be used to the complicated nonlinear system by the fast computer systems; however, the use of symbolic computation in nonlinear control, due to the very increase of the sizes of higher order derivatives, is limited (de Jager, 1995; Röbenack, 2005; Griewank and Walther, 2008). The computational effort basically increases exponentially for higher order derivatives. Therefore, the numeric differentiation due to cancellation and truncation errors is not applicable. These derivatives can efficiently be computed using an alternative differentiation technique called algorithmic differentiation (Kwatny and Blankenship, 2000; Röbenack, 2005; Röbenack, 2007; Griewank et al., 2000; Röbenack and Reinschke, 2004).

Higher order Lie derivatives in nonlinear control can easily be calculated by algorithmic differentiation using Taylor arithmetic. This approach can be used to compute the mixed Lie derivatives. In the present study, the Lie derivatives are defined and algorithmic differentiation is employed to compute the mixed Lie derivatives.

Lie Derivatives
Consider a vector field \( f \colon M \to \mathbb{R}^n \) and the associated flow \( \varphi_t \). Lie derivatives of a scalar field \( h \colon M \to \mathbb{R} \) are given by:

\[
L^k_{\varphi} h(p) = \lim_{\tau \to 0} \frac{h(\varphi_{\tau}(p)) - h(p)}{\tau}, \quad L^k_{\varphi} h(p) = \frac{\partial^k}{\partial t^k} h(\varphi_t(p)) \bigg|_{t=0}, \quad L^0 h(x) = h(x)
\]

(1)

For a further vector field \( g \colon M \to \mathbb{R}^n \), the mixed Lie derivatives are obtained as follows:

\[
L^k_{\varphi} L^l_{\varphi} h(p) = \lim_{\tau \to 0} \frac{h(\varphi_{\tau}(p)) - h(p)}{\tau}, \quad L^k_{\varphi} L^l_{\varphi} h(x) = \frac{\partial^{k+l}}{\partial t^k \partial s^l} h(\varphi_t(\varphi_s(p))) \bigg|_{t=s=0}
\]

(2)

Consider an unforced state-space system:

\[
x = f(x), \quad y = h(x), \quad x(0) = p \in M
\]

(3)
The time derivatives of the output can be written as follows:

\[ y(0) = L_1 h(p), \quad y(0) = L_2 h(p), \ldots, \quad y^{(k)}(0) = L_k h(p) \quad (4) \]

The Lie derivatives can be written as follows:

\[ y(t) = \sum_{k=0}^{\infty} L_k h(x_0) \frac{t^k}{k!} \quad (5) \]

This expansion is called Lie series (Röbenack, 2007). The mixed Lie derivatives can be modelled by a control system of the following form:

\[ x = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad y = h(x), \quad x(0) = x_0 \in M \quad (6) \]

With additional vector fields \( g_1, \ldots, g_m \): \( M \to \mathbb{R}^n \) and the control inputs \( u_1, \ldots, u_m \). There are several approaches to extend the concept of the Lie series (5) to control system (6). Table 1 shows a straightforward implementation in Maple using the linear algebra package linalg for higher order Lie derivatives.

**Table 1: Simply Maple Implementation of Lie Derivatives**

<table>
<thead>
<tr>
<th>With (linalg);</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lie derivative = process(f,h,x);</td>
</tr>
<tr>
<td>Multiply (jacobian (h, x), f)</td>
</tr>
<tr>
<td>end process</td>
</tr>
</tbody>
</table>

**Computation of Lie Derivatives**

In this work, the Lie derivatives are computed by algorithmic differentiation. For the computation of higher order derivatives the concept of Taylor arithmetic was introduced. The Taylor series are given by:

\[ x(t) = x_0 + x_1 t + x_2 t^2 + \ldots + x_d t^d + \mathcal{O}(t^{d+1}) \quad (7) \]

with \( x_0 \in M \in \mathbb{R}^n, x_1, \ldots, x_d \in \mathbb{R}^n \) into a curve as follows:

\[ z(t) = f(x(t)) = z_0 + z_1 t + z_2 t^2 + \ldots + z_d t^d + \mathcal{O}(t^{d+1}) \quad (8) \]

The first Taylor coefficients of (8) are given by:

\[ z_0 = F(x_0), \quad z_1 = F'(x_0)x_1 \quad (9) \]

The next Taylor coefficients can be written as follows:

\[ z_2 = F'(x_0)x_2 + \frac{1}{2} F''(x_0)x_1x_1, \]

\[ z_3 = F'(x_0)x_3 + F'(x_0)x_1x_2 + \frac{1}{6} F'''(x_0)x_1x_1x_1, \quad (10) \]

\[ z_4 = F'(x_0)x_4 + F'(x_0)x_1x_3 + \frac{1}{2} F'(x_0)x_2x_2 + \frac{1}{2} F''(x_0)x_1x_1x_2 + \frac{1}{24} F^4(x_0)x_1x_1x_1. \]

The Taylor coefficients for elementary functions are given in Table 2.

Although, in theory, derivatives could be calculated from linear combinations of the \( z \) vectors for a set of appropriate Taylor coefficients. However, this method is not applicable for nonlinear control systems. By consideration of the following function:

\[ x(t) = x_0 + w_1 t + w_2 t^2 + w_3 t^3 + \ldots \quad (11) \]

The mixed Lie derivative along the vector fields can be obtained by the algorithmic differentiation tool ADOL-C as follows:
The nonlinear model is computed by the computer algebra system Maple 10. The time of calculation of the Lie derivatives with Taylor series and modified function for the above nonlinear model is illustrated in Figure. As shown, the calculation of lower order Lie derivatives is faster with the optimized function and the optimization of the code generation reduces the run time significantly. The similar trend is obtained by Röbenack (2008) for the nonlinear benchmark problem of a chemical reactor.

Example
Consider a perfectly mixed, continuous stirred tank reactor (CSTR). The reaction of $2A \rightarrow B$ is taking place in the reactor. A material balance on species $A$, leads to the following nonlinear model.

$$V \frac{dC_A}{dt} = FC_{A,in} - FC_A - 2kVC_A^2$$

(13)

where, $V$ is the reactor volume, $C_{A,in}$ is the concentration of $A$ in the feeding stream and $F$ is the flow rate. System can be rewritten in dimensionless variables as follows:

$$\frac{dx}{dr} = -(1 + 2a)x + au - ux - ax^2$$

(14)

The above system is suitable for evaluating the relative merit of the Taylor method and modified Taylor method.

Table 2: Computation of Taylor Coefficients

<table>
<thead>
<tr>
<th>Operations</th>
<th>Taylor Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = x \pm y$</td>
<td>$z_k = x_k + y_k$</td>
</tr>
<tr>
<td>$z = xy$</td>
<td>$z_k = \sum_{i=0}^{k} x_i y_{k-i}$</td>
</tr>
<tr>
<td>$z = x/y$</td>
<td>$z_k = (x_k - \sum_{i=1}^{k} y_i z_{k-i}) / y_0, y_0 \neq 0$</td>
</tr>
<tr>
<td>$z = x^{0.5}$</td>
<td>$z_k = (x_k - \sum_{i=1}^{k-1} y_i z_{k-i}) / 2z_0, k \geq 1, z_0 = \sqrt{x_0}$</td>
</tr>
<tr>
<td>$z = \exp(x)$</td>
<td>$z_k = (\sum_{i=0}^{k-1} (k-i)z_i x_{k-i}) / k_0, k \geq 1, z_0 = \exp(x_0)$</td>
</tr>
<tr>
<td>$z = \ln(x)$</td>
<td>$z_k = (x_k - \sum_{i=1}^{k-1} iz_i x_{k-i}) / k), z_0, k \geq 1, z_0 = \ln(x_0)$</td>
</tr>
</tbody>
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Conclusion
In this work, the multiple Lie derivatives in nonlinear control system was computed. The first order derivatives was computed symbolically by computer algebra system. For higher order Lie derivatives, the use of a symbolic computation is impossible. Therefore, the algorithmic differentiation tools based on Taylor arithmetic is used for calculation of mixed Lie derivatives. Moreover, the modified function is applied to increase the speed of algorithmic differentiation to compute Taylor coefficients.

REFERENCES
Gómez JC (1994). Using symbolic computation for the computer aided design of nonlinear (adaptive) control systems. In: 14th IMACS World Congress on Computational and Applied Mathematics, Atlanta, USA.

Figure 1: CPU Time for the Evaluation of Lie Derivatives for Nonlinear Control System Using Taylor Series and Optimized Function


