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***Research Article***

# MULTI-STEPSTOCHASTIC DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVINGOPTIMAL CONTROL PROBLEMS

***Soleimanivareki M.[1, 2], Fakharzadeh A.J.[2] and Hesamaeddini E.[2]**
[1]*Department of Math, Islamic Azad University, Ayatollah Amoli Branch, Amol, Iran*
[2]*Department of Math, Faculty of Basic Sciences, Shiraz University of Technology, Shiraz, Iran*
**Author for Correspondence*

## ABSTRACT
In classical methods for solving stochastic optimal control problems, the main attempts are focused to determine the solution by introducing the value function via dynamic programming and Hamilton-Jacobi-Bellman equation. But in general, the determination of solution in a closed form is not simple. For this necessity and to find of an appropriate optimal trajectory and control, in this article we introduce the multi-step numerical-analytical method to solve stochastic optimal control problems (SOCPs). In new our approach, the differential transformation method is extended to stochastic type and regarding the properties of Brownian motions in $L^2-$spaces, the approximated optimal strategies for constructing a general continuous–time finite-horizon SOCPs are illustrated. An applicable example in management science is also presented to show the ability and efficiency of the new method especially in comparison.

***Keywords:*** *Stochastic Optimal Control Problems, Multi-step Stochastic Differential Transformation Method, Hamilton-Jacobi-Bellman equation, Karhunen-Loeve Expansion, Merton Problem.*

***Mathematics Subject Classification:*** 93E20,93E35.

## INTRODUCTION
SOCPs frequently occur in many branches of sciences, especially in economics and finance. Stochastic differential equations have become the standard models for financial quantities such as asset prices, interest rates, and their derivatives (Oksendal, 2003; Steele, 2001). By regards the fact that dynamic programming is one of the importance techniques for solving optimal control problems, as a usual manner, it has been used for SOCPs as well (Kirk, 2004). In the case of continuous optimal control problems, dynamic programming technique reduces the optimal control problem to solve a partial differential equation (the Hamilton-Jacobi-Bellman (HJB) equation). Almost in all (just a few special cases) these equations are difficult to be solved analytically (Kirk, 2004). Even several approaches for solving the corresponding HJB equation are presented in (Yong and Zhou, 1999; Kushner and Dupuis, 2001). But only a few classes of SOCPs admit analytical solutions for the value function in this manner. Besides, Finite state Markov chain approximation and finite differences are two classical approaches in solving SOCPs numerically (Oksendal, 2003), but, analytical and analytical- numerical methods are superior compared to numerical methods due to the determination of closed form solution.

One of the effective methods for solving differential equations is differential transformation method (DTM) which was originally introduced by Zhou (1986). This method obtains an analytical solution in the form of a polynomial on based of Taylor series for differential equations (Abazari and Abazari, 2010), and has be applied for solving deterministic optimal control problems (see Fakharzadeh and Hashemi, 2012; Hesameddini *et al.*, 2012). In multi-step DTM, a modification of DTM and the convergence of the obtained solution series are improved. In 2010, Gokdogan and Merdan, was proposed an efficient and fast approach for the multi-step DTM, a reliable modification of the DTM that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions. For stochastic calculus, the DTM is developed to solve random differential equations (Villafuerte and Chen-Charpentier, 2012).

The purpose of this paper is to extend the application of multi-step DTM for obtaining approximated analytical solution of SOCPs. For this aim, we use HJB equation and multi-step stochastic differential

transform method (MSDTM) to determinate the approximated optimal trajectory and optimal control of SOCPs based on the Brownian motion properties in $L^2-$space. This procedure is presented in this paper areas follows.

First, we present preliminaries and basic definitions that will be used. In section 3, we discussed the SOCP's and its necessary and sufficient optimality conditions. The new multi-step method is explained in section 4 and then is proved existence of solution and it's convergence. In order to demonstrate the application and efficiency of the new method, section 5 is devoted to describe and solve one of the famous and useful problems in management, namely Merton problem, by applying MSDTM and the results are compared with the other methods. Finally, a brief conclusion is presented in section 6.

***Some Preliminaries and Basic Definition***

In order to introduce the tools for the next discussions, in this section we remind some necessary basic definitions and properties which mostly can be found in Steele (2001) and Soong (1973).

***Definition 1.*** For a given probability space $(\Omega, F, P)$, a real random variable $X$ which is defined on this space and satisfying the condition $E(X^2) < +\infty$, is called a second order random variable (2-r.v.); here $E$ denotes the expectation operator. The space $L_2$ of all the 2-r.v.'s endowed with the norm $\|X\|_2 = \sqrt{E(X^2)}$, is a Banach space (Soong, 1973).

***Definition 2.*** In the probability space $(\Omega, F, P)$, a stochastic process $\{X(t): t \in I\}$ where $I$ is a closed interval in the real line $\Re$, is called a second order stochastic process (2-s.p.) if for each $t \in I, X(t)$ is a 2-r.v. A sequence of 2-r.v.'s $\{X_n : n \geq 0\}$ is mean square convergent in $L_2$ to a 2-r.v. $X$ as $n \to \infty$ if $\lim_{n\to\infty} \|X_n - X\|_2 = 0.$ Also, if we have a limiting condition such that $\lim_{\Delta t \to 0} \left\| \frac{X(t+\Delta t) - X(t)}{\Delta t} - \dot{X}(t) \right\|_2 = 0$ then, 2-s.p $\{\dot{X}(t): t \in I\}$ is the mean square derivative of $\{X(t): t \in I\}$.

***Remark 1:*** regard to this fact that mean square norm is not sub-multiplicative; this fact motivates the development of the mean fourth calculus, see (Villafuerte *et al.*, 2010). Thus a r.v.'s $X$ such that $E(X^4) < +\infty$ is fourth order random variables which will be denoted by 4-r.v.'s. The set $L_4$ with norm $\|X\|_4 = \sqrt{E(X^4)}$ is also a Banach space (Soong, 1973). For more details about the properties and theorems of random calculus, one can see Villafuerte *et al.*, (2010) and the references cited in Soong (1973).

***Definition 3.*** Let $k \in Z$ and assume that the 4-s.p. $\{v(t): t \in I\}$, has mean fourth derivative of order $k$ at $t \in I$ which is denoted by $v^{(k)}(t)$. The random differential transform of the process $v(t)$ is defined as:

$$V(k) = \frac{1}{k!}\left[\frac{d^k(v(t))}{dt^k}\right]_{t=t_0}, \tag{1.2}$$

where $V$ is the second process transformed and $\frac{d}{dt}$ denotes the mean square derivative. The inverse transform of $V$ is defined as:

$$v(t) = \sum_{k=0}^{\infty} V(k)(t-t_0)^k, \tag{2.2}$$

here, formally it is assumed that the series (2.2) is uniformly mean fourth convergent in any closed interval into the domain of convergence and as a result, (2.1) and (2.2) are well defined (Villafuerte and

Chen-Charpentier, 2012). For implementation purposes, the function *v(t)* is expressed by a finite series and Eq. (2.2) can be written as $v(t) \approx \sum_{k=0}^{N} V(k)(t-t_0)^k$, that $N$ is decided by the convergence of natural frequency.

***Theorem 1.*** Assume that the $k-$order mean fourth derivatives of the 4-s.p.'s $\{v_1(t): t \in I\}$ and $\{v_2(t): t \in I\}$ exists at $t \in I$, then the following results hold (Villafuerte and Chen-Charpentier, 2012):

(i) If $v(t) = v_1(t) \pm v_2(t)$, then the random differential transform of $v(t)$ is given by $V(k) = V_1(k) \pm V_2(k)$.

(ii) Let $A$ be a 4-r.v., if $v(t) = Av_1(t)$ then the random differential transform of $v(t)$ is given by $V(k) = AV_1(k)$.

(iii) If $v(t) = \dfrac{d^m(v_1(t))}{dt^m}$ then the random differential transform of $v(t)$ is given by $V(k) = (k+1)(k+2)\cdots(k+m)V_1(k+m)$.

(iv) If $v(t) = v_1(t)v_2(t)$ then the random differential transform of $v(t)$ is given by $V(k) = \sum_{l=0}^{k} V_1(k)V_2(k-l)$.

***Definition 4.*** Standard Brownian motion was named after Robert Brown observed it in pollens of grains in water. The process was described mathematically by Norbert Wiener, and it is thus also called Wiener processes. Mathematically a $1-$dimensional Standard Brownian motion $B(t)$ is defined by the following properties:

(i) $P(B(0) = 0) = 1$;

(ii) The probability that a randomly generated Brownian path to be continuous, is one;

(iii) The path increments are independent Gaussian, zero mean, with variance equal to the temporal extension of the increment.

Wiener showed that, there is a stochastic process that does not violate the axioms of probability theory and satisfies the 3 aforementioned properties (Oksendal, 2003). Furthermore, a stochastic process $B(.) = (B_1(.), B_2(.), \cdots, B_n(.))$ is a $n-$dimensional Brownian motion provided

(i) For each $k = 1, 2, \cdots, n, B_k(.)$ is a $1-$dimensional Brownian motion;

(ii) The $\sigma-$algebras $\mathrm{B}^k := F(B_k(t): t \geq 0)$ are independent, $k = 1, 2, \cdots, n$.

***Remark 2:*** Random differential equations are defined as differential equations involving random elements. Accordingly, we can consider 3 main parts in each random differential equation:

a) Random Initial Conditions,
b) Random Inhomogeneous Parts,
c) Random coefficients.

For example, a type of the random Riccati differential equation is introduced as (Villafuerte and Chen-Charpentier, 2012):

$$\frac{dx(t)}{dt} = \lambda^2 x(t);\ x(0) = x_0,\ t \in [0,1],$$

Where $\lambda$ and $x_0$ are independent 4-r.v's. In engineering, a stochastic differential equation (SDE) is a differential equation driven by "white noise":

$$\frac{dx(t)}{dt} = a(x(t),t) + \sigma(x(t),t)w(t),$$

where $w(t)$ is "white noise". Formally, white noise is a derivative of Brownian motion:

$$\frac{dB(t)}{dt} = w(t).$$

That is, it is a Gaussian process with an autocorrelation function equal to Dirac delta function (Soong, 1973). In finance, it is more common to use the following notation:

$$dx(t) = a(x(t),t)dt + \sigma(x(t),t)dB(t), \tag{3.2}$$

to represent the SDEs (Oksendal, 2003). In this case, $dB(t)$ is interpreted as an increment of Brownian motion, which is normally distributed with mean zero and variance $dt$; that is:

$$dB(t) = B(t,dt) - B(t) \approx N(0,dt)$$

A solution of a SDE is a collection of sample paths of a stochastic process, plus probabilistic information as to the likelihoods of the various paths. For more details and theorems of SDEs especially existence and uniqueness of solution theorems, one can see (Oksendal, 2003).

The popularity of SDE in control and filtering applications is adequate, because it is a natural random extension of the powerful state space method in classical optimal control theory. Moreover, although white noise is a mathematical trick, it approaches the behavior of numerous of important noise processes in electrical systems (Soong, 1973).

***Problem Statement***

In this section, we consider finite-time homogeneous SOCPs of the type

$$Sup_{u\in U}\ E(\int_0^T e^{-\beta t}F(x(t),u(t))dt + g(x(T))),$$

$$S.to: \qquad dx(t) = f(x(t),u(t)) + \sigma(x(t),u(t))dB(t);\ x(0) = x_0, \tag{1.3}$$

where $u \in U \subseteq R^n$ denotes the set of admissible controls and $x \in \mathrm{X} \subseteq R^m$ is a state vector. Furthermore, $F : \mathrm{X}\times U \to R$ is the utility function and $\beta \geq 0$ is the discount rate.

We restrict this problem to the one-dimensional SOCPs for simplicity of the exposition. Our approach to solve (1.3) is to write down the HJB equation and use the problem conditions to identify a value function. Hence, our goal is to find an optimal control $u^*(.)$ such that:

(i) It is a mapping of $[t,T]$ into $U$, that for each time $t \leq s \leq T$, $u(s)$ depends only on $s$ and observations of $x(\tau)$ for $t \leq \tau \leq s$.

(ii) $P_{x,t}[u^*(.)] = \max_{u(.)} P_{x,t}[u(.)] = \max_{u(.)} E\{\int_t^T e^{-\beta t}F(x(s),u(t))ds + g(x(T))\}.$

For this purpose, we firstly define the value function $v(x,t) := \sup_{u(.)} P_{x,t}[u(.)]$. If $v \in C^1$ then it must satisfy the following partial differential equation that was named stochastic HJB equation (for more details see Kushner and Dupuis, 2001; Ewald and Wang, 2011):

$$v_t(x,t) + \max_{u\in U}\{f(x,u)\nabla_x v(x,t) + \frac{\sigma^2(x,u)}{2}\Delta v(x,t) + F(x,u)\} = 0;\ v(x,T) = g(x). \tag{2.3}$$

We re-emphasize that, the purpose is to obtain feedback control rule such that it is raised above two features. For this work, similar to deterministic case of OCPs (see Kirk, 2004; Kushner and Dupuis, 2001), we write all steps for determine of control function as the step by step. Our algorithm that will be used to find optimal control for this problem is as follows:

***Algorithm:***

***Step1.*** For each given point $(x,t)$ compute a value for $u \in U$ such that (3.3) attains its maximum.

$$f(x,u)\nabla_x v(x,t)+\frac{\sigma^2(x,u)}{2}\Delta v(x,t)+F(x,u). \qquad (3.3)$$

It should be explained that the maximum of (3.3) is determined in terms of $x, v, v_t, v_x$ and $v_{xx}$ variables, via the first or second derivative test.

***Step 2.*** We make the sophisticated guess of value function via utility function's form and other information of the problems. It is clear that, the value function is a function of $x$ and $t$ variables.

***Step 3.*** With substitution the sophisticated guess in the result of step1, the feedback optimal control can be computed.

***Step 4.*** Now, by substituting the optimal control from step3 would be only a SDE in terms of variables $x$ and $t$:

$$dx^*(s)=f(x^*(s),u(x^*(s)))ds+\sigma(x^*(s),u(x^*(s)))dB(s);\ x^*(t)=x; \qquad (4.3)$$

then, by solving (4.3)the optimal trajectory can be determined.

***Multi-step Stochastic Differential Transformation Method (MSDTM)***

In section 3, SOCPs were introduced by(1.3) and the HJB approach was remained for solving of these problems. Now, in this section first we introduce multi-step stochastic differential transform method for solving this class of optimal control problems and then examine existence and uniqueness's solution of the new approach.

***Introducing New Approach***

For the stochastic differential equation (4.3), we assume that all the involved stochastic quantities take values in$L_2$-space and also all the stochastic operations are in the mean square sense. We note that $B(t)$ is a Gaussian process with mean zero and it is also a mean fourth continuous process (Stefano, 2008). The Brownian motion $B(t)$ has trajectories belong to $L^2([0,T])$ for almost all events. In this space Karhunen-Leove expansion (Stefano, 2008) for Brownian motion takes the form

$$B(t)=B(t,\omega)=\sum_{i=0}^{\infty} z_i(\omega)\varphi_i(t),\ 0\le t\le T,$$

With

$$\varphi_i(t)=\frac{2\sqrt{2T}}{(2i+1)\pi}\sin\left(\frac{(2i+1)\pi t}{2T}\right)$$

where the functions $\varphi_i(t)'s$ form a basis of orthogonal functions and $\{z_i\}$ is a sequence of independent and identically distributed Gaussian random variables (Stefano, 2008). This approach is also quite powerful for simulating paths of processes without independent increments of Brownian motion. By substituting finite terms of Karhuneun-Loeve expansion in (3.2), we have:

$$dx(t)=a(x,t)dt+\sigma(x,t)d\left(\sum_{i=0}^{M} z_i\varphi_i(t)\right),\ x(0)=x_0. \qquad (1.4)$$

Applying the Random differential transformation method from (1.2) and using the properties (I)-(IV) in Theorem 1, one can transfer (1.4) in to the following algebraic equation:

$$(k+1)X(k+1)=A(k)+\sum_{i=0}^{m}\sum_{j=0}^{k} z_i\Psi_i(k)\Sigma(k-j);\ X(0)=x_0, \qquad (2.4)$$

where $X(k), A(k), \Sigma(k)$ and $\Psi(k)$ are the transformed processes of $x, a, \sigma$ and the derivative of $\varphi_i$ respectively. In order to simulate $z_i$, for instance, one can use the maple random variable generator (*Random Tools Flavor: distribution)*.

We remind that, *[0, T]* is the interval which we want to find the solution of (1.4) over *t* and we apply a multi-step approach for ensuring validity of the mentioned approximations for large $T$. In our new

approach we assume that the interval *[0, T]* is divided into *M* subintervals $[t_{i-1}, t_i]$, $i = 1,2,..., M$ with equal step length. Thus, first, we use the SDTM to Eq. (1.4) over the interval $[0, t_1]$; then at each subinterval $[t_{i-1}, t_i]$, for $i \geq 2$, the SDTM is applied to Eq. (1.4) over the interval $[t_{i-1}, t_i]$, where $t_0$ is replaced by $t_{i-1}$. It is necessary that refer to the initial conditions of the above approach are as: $x_i(t_{i-1}) = x_{i-1}(t_{i-1})$ for $i = 2,3,...,M$. Therefore, the process is repeated to generate a sequence of approximate solutions $x_i(t)$, $i = 1,2,...,M$. In fact, the MSDTM obtains the following solution as a piecewise polynomial on $[0,T]$:

$$x(t) = \begin{cases} x_1(t), & t \in [0, t_1], \\ x_2(t), & t \in [t_1, t_2], \\ \vdots & \\ x_M(t), & t \in [t_{M-1}, t_M], \end{cases} \tag{3.4}$$

For more details and application of MSDTM, one can see Fakharzadeh *et al.*, (2015).

***Existence and Uniqueness of the MSDTM Solution***

In previous section, we introduced a new multi-step approach for solving SOCPs. In this manner, we achieved to a numerical-analytical approximation of the optimal trajectory of (1.3). For warranty and acquisitions resulting of the proposed method it is required to show the existence of solution; thus, the existence and uniqueness of the solution of new obtained approach will examine with remarking some theorems in the following section. These theorems together guarantee the convergence and achievement of the obtained result with SDTM for some classes of SOCPs (Villafuerte *et al.*, 2010).

***Theorem 2:*** Consider the problem

$$\begin{cases} \dot{X}(t) = P_n(t)X(t) + Q(t); & P_n(t) = \sum_{i=1}^{n} a_i t^i, \quad t \in T \\ X(0) = X_0, \end{cases} \tag{4.4}$$

Assume that the random variables $a_i$ satisfy the condition

$$E\left[\left|a_j^m\right|\right] \leq KM^m < \infty, \qquad \forall m \geq 0, \qquad j = 0,1,...,n,$$

Where $X_0$ is 4-r.v and $Q(t)$ is a 2-th analytic stochastic process. Then there exists a solution of the from

$$X(t) = \sum_{k=0}^{\infty} \hat{X}(k) t^k; \qquad \hat{X}(k+1) = \frac{1}{k+1}\left(\sum_{r=0}^{k} \hat{P}_n(r)\hat{X}(k-r) + \hat{B}(k)\right); \; |t| < c = \min\left(\rho, \frac{1}{n+1}\right).$$

Proof: See (Villafuerte and Chen-Charpentier, 2012; Villafuerte *et al.*, 2010).

***Remark 3:*** by the definition of truncated process $X_N(t) = \sum_{k=0}^{N} \hat{X}(k) t^k$, we have:

$$\begin{aligned} &a) \quad E\left[X_N(t)\right] = \sum_{k=0}^{N} E\left[\hat{X}(k)\right] t^k; \\ &b) \quad E\left[\left(X_N(t)\right)^2\right] = \sum_{k=0}^{N} E\left[\left(\hat{X}(k)\right)^2\right] t^{2k} + 2\sum_{k=1}^{N}\sum_{l=0}^{k-1} E\left[\left(\hat{X}(k)\hat{X}(l)\right)\right] t^{k+l}. \end{aligned} \tag{5.4}$$

Moreover, if a stochastic process $X_N(t)$ is convergent in $L_2[0,T]$ to $X(t)$ in mean square sense then, the expectation of $X_N(t)$ will be convergent to $X(t)$.(indeed this theorems proves the convergence of our approach).

***Theorem 3:*** Consider the random initial value problem (6.4):

$$\begin{cases} \dot{X}(t) = F(X(t),t), & 0 < t < T, \\ X(0) = x_0, \end{cases} \tag{6.4}$$

where $x_0$ is a second order random variable and the unknown $X(t)$ as well as the second member. If $F: S \times T \to L_2$ is continuous and satisfies the m.s. in the following Lipchitz condition

$$\|F(X,t) - F(Y,t)\|_2 \le K(t)\|X - Y\|_2,$$

where $\int_{t_0}^{t_e} k(t)dt \le \infty,$ there exists a unique m.s. solution for any initial condition $x_0 \in L_2$.

Proof: See (El-Tawil and Tolba, 2013).

***Proposition 1:*** Consider the problem (4.4) and assume that all of the conditions of theorem2 are satisfied. Also by defining $F(X(t),t) = P_n(t)X(t) + Q(t)$, we assume $F: S \times T \to L_2$ is continuous and satisfies the m.s. Lipchitz condition. Then, there exists a unique m.s. solution for any initial condition $x_0 \in L_2$.

***Sketch of the Proof:*** The existence of the presented solution for (3.4) obtained by SDTM is proved by Theorem 2. Note that even MSDTM is similar to SDTM, but in MSDTM has extra action where we have encountered to finite approximation based on orthogonal polynomials for white noise and we have applied a piecewise polynomial for trajectory.

Thus, convergence the obtained solution of (3.4) by MSDTM to the exact solution is proved; now, according to Theorem3 this solution is unique.

***Proposition 2:*** The convergence of MSDTM is faster than SDTM.

***Proof:*** In above remark, it is claimed that the convergence order of MSDTM is greater than SDTM. This fact is easy to prove; we remind that the stochastic differential transform of $\exp(X(t))$ has the form $\sum_{n=0}^{\infty} \frac{(X(t))^n}{n!}$ (Villafuerte *et al.*, 2010). Thus, by defining $sin(X(t)) = \frac{1}{2}(e^{iX(t)} - e^{-iX(t)})$ and $\cos(X(t)) = \frac{1}{2}(e^{iX(t)} + e^{-iX(t)})$, one can determine the stochastic differential transforms of $sin(X(t))$ and $\cos(X(t))$.

Furthermore, due to a range partitioned as $\{0, t_1, t_2, ..., t_M\}$, for each $t \in [0,T]$ there is a number *m* such that $t \in [t_m, t_{m+1})$ where $m = 0,1,...,M-1$. In SDTM we have $X(t) = \sum_{n=0}^{N} a_n (t - t_0)^n$; but in MSDTM the approximated trajectory is a piecewise polynomial $X(t) = \sum_{m=0}^{M-1} \chi_{[t_m, t_{m+1}]} X_m(t)$ *where* $X_m(t) = \sum_{n=0}^{N} a_{mn}(t - t_{m-1})^n$; we will show that:

$$\left|a_{mN}(t-t_{m-1})^N\right| \le \left|a_N(t-t_0)^N\right|. \tag{7.4}$$

Let $a_{mN} = \dfrac{f^{(N)}(t_{m-1})}{N!}$ and $a_N = \dfrac{f^{(N)}(t_0)}{N!}$; therefore, (7.4) turn into (8.4) below:

$$\left|(t-t_{m-1})^N f^{(N)}(t_{m-1})\right| \le \left|(t-t_0)^N f^{(N)}(t_0)\right|. \tag{8.4}$$

If we take $b = f^{(N)}(t_{m-1})$ and $a = f^{(N)}(t_0)$, then it is necessary to show that $\left|\dfrac{b}{a}\right| \le \left|\dfrac{(t-t_0)}{(t-t_{m-1})}\right|^N$. Note that, $t_M - t_0 = M\delta$ and $(t-t_{m-1}) < \delta$ thus, we have $(m-1)^N > \left|\dfrac{a}{b}\right|$. Therefore, to establish (7.4), it is sufficient to choose *m* greater than $1+\sqrt[N]{\dfrac{a}{b}}$.

***A Case Study and Simulation: Merton's Portfolio Problem***

In order to explain the application of the mentioned method for solving SOCPs, as an example, we solve the useful economic problem which is worked out by Merton in (Merton, 1971). Merton's portfolio problem is a famous problem in continuous-time finance. This problem was formulated and solved by Merton in 1969, for both finite lifetimes and for the infinite case. In Merton's model, we have the option of investing some of our wealth in either a risk-free bond or a risky stock. We also intend to consume some of our wealth as time evolves. This problem assumes that, an investor has a portfolio consisting of two assets, risk-free and risky, that the price $b(t)$ per share of the bond changes according to $db = rbdt$ whereas for the stock (the stock is changing according to a stochastic differential equation) we have $dS = S(Rdt + \sigma dB)$, where $r, R$ and $\sigma$ are constants, with $R > r > 0,\ \sigma \ne 0$. Let $x(t)$ denote the investor's wealth at time $t$ and $u_1(t)$ be the fraction of wealth in the risky asset and $u_2(t)$ be the consumption rate; thus, $u(t) = (u_1(t), u_2(t))$ is control variable and attains its values in $U = [0,1]\times[0,+\infty)$. In this manner, the total wealth evolves as:

$$dx(t) = (1-u_1(t))x(t)\frac{db}{b} + u_1(t)x(t)\frac{dS}{S} - u_2(t)dt.$$

Therefore:

$$dx(t) = r(1-u_1(t))x(t)dt + u_1(t)x(t)\{Rdt + \sigma dB(t)\} - u_2(t)dt,\ x(0) = x_0. \tag{1.5}$$

We stop the process if the wealth reaches zero (bankruptcy); we also assume that the running cost is $F(x(t), u(t)) = l(u_2(t))$, where it is the utility of consuming at rate $u_2(t) > 0$. Thus, the problem is to maximize the total expected utility, discounted at rate $\beta > 0$:

$$E\left(\int_0^\tau e^{-\beta t} l(u_2(s))ds\right),$$

where $\tau$ denotes the random first time $x(.)$ leaves $Q = \{(x,t) : 0 \le t \le T, x \ge 0\}$. In (El-Tawil and Tolba, 2013) the HJB equation for this SOCP is determined as:

$$v_t + \max_{\substack{u_2 \ge 0 \\ 0 \le u_1 \le 1}} \left\{\frac{(u_1 x\sigma)^2}{2} v_{xx} + (r(1-u_1)x + Rxu_1 - u_2)v_x + e^{-\beta t} l(u_2)\right\} = 0, \tag{2.5}$$

with boundary conditions $v(0,t) = 0$ and $v(x,T) = 0$. Now we write the steps of algorithm for solving this problem which is introduced in section 3, as follows:

***Step1.***By computing the maximum of $\frac{(u_1 x\sigma)^2}{2} v_{xx} + (r(1-u_1)x + Rxu_1 - u_2)v_x + e^{-\beta t} l(u_2)$ respect to $u$, via the first derivative test, we find $u_1^* = \frac{-(R-r)v_x}{\sigma^2 x v_{xx}}$ and $l'(u_2^*) = e^{\beta t} v_x$, provided that the constraints $0 \le u_1^* \le 1$ and $0 \le u_2^*$ are valid.

***Step2.*** We assume the utility function has the form $l(u_2) = u_2^{\gamma}, 0 < \gamma < 1.$ Next we guess that the value function has the form $v(x,t) = g(t)x^{\gamma}$ in which, $g(t)$ to be determined in the next step.

***Step3.*** Optimal control in feedback form can be computed as (Fleming and Soner, 2006):

$$u_1^*(t) = \frac{R-r}{\sigma^2(1-\gamma)},\ u_2^*(t) = [e^{\beta t} g(t)]^{\frac{1}{\gamma-1}} x(t), \tag{3.5}$$

where

$$g(t) = e^{-\beta t}[\frac{1-\gamma}{\beta-\lambda\gamma}(1-\exp(\frac{-(\beta-\lambda\gamma)(T-t)}{1-\gamma})]^{1-\gamma}, \lambda := \frac{(R-r)^2}{2\sigma^2(1-\gamma)} + r.$$

It should be explained that, if $(R-r) \le \sigma^2(1-\gamma)$ then, the mentioned in conditions of Step1 are quarantined (Fleming and Soner, 2006).

***Step4.*** By substituting (3.5) in (1.5), the following stochastic differential equation is obtained:

$$dx(t) = \{r(1-\frac{R-r}{\sigma^2(1-\gamma)}) + \frac{R(R-r)}{\sigma^2(1-\gamma)} - [\frac{1-\gamma}{\beta-\lambda\gamma}(1-\mathrm{e}^{\frac{-(\beta-\lambda\gamma)(T-t)}{1-\gamma}})]^{-1}\}x(t)dt + \frac{R-r}{\sigma(1-\gamma)}x(t)dB(t). \tag{4.5}$$

Then, an approximated analytical wealth is presented by solving (4.5) via MSDTM; an hence one could find an optimal feedback control.

In (Krawczyk, 1999), it was supposed that an investor with wealth of $x_0 = 10^5$ unit wants to maximize their satisfaction during the coming $T = 10$ years. Also, the considered parameters in this article are $r = 0.05, R = 0.11, \sigma = 0.4$ and $\beta = 0.11.$ In this case, by using MSDTM, we solved the problem. Figure 1 presents the approximated optimal controls of MSDTM and the expected wealth is also shown in Figure 2 for 5 times random run. We have also run the new method for this problem fifty times randomly and thus the average wealth of them is plotted in Figure 3. We remind that, one can solve (1.5) without white noise and indicates similarity of the obtained solution to this figure; this fact confirms the robustness of our new approach. Also the realization of five wealth and strategy, which correspond to five random samples, are presented in Figure 3; By the way, the average total discounted utility of these 5 portfolios was obtained as 727.06 with this method. It is clear that, the investor's expected discount total utility will be $v(x_0, 0) = g(0)\sqrt{10^5} = 723.09.$ This value was computed as 751, and thus the obtained value of MSDTM is nearer the expected discount total utility. These obtained results are more accuracy than that was reported in (Krawczyk, 1999); this fact is shown the efficiency of the new method.

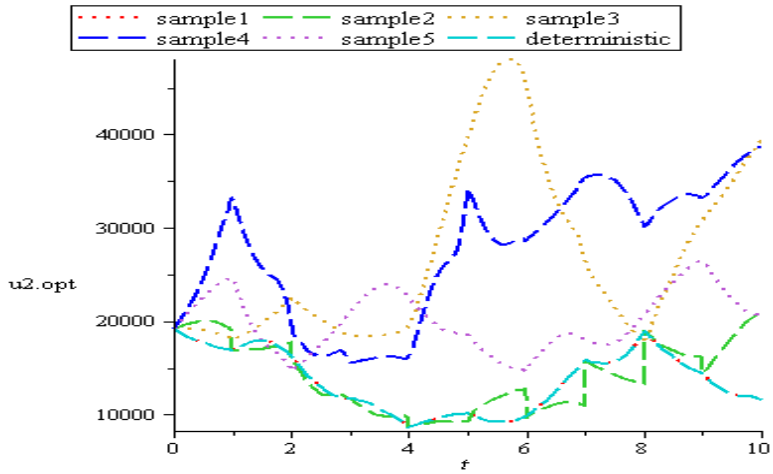


**Figure 1: The optimal strategies for 5 samples**

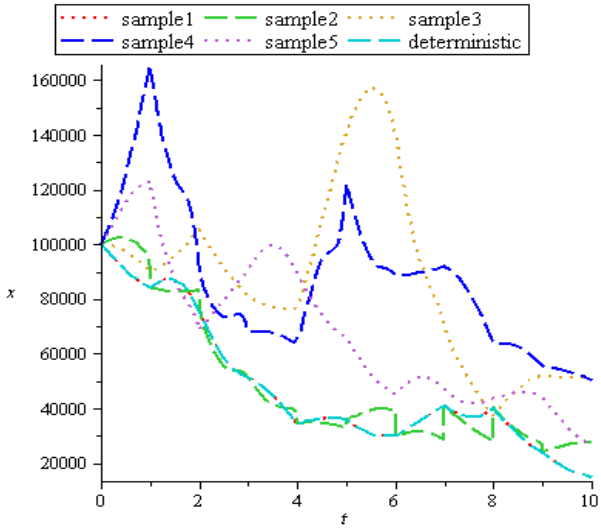


**Figure 2: The optimal wealth for 5 samples**

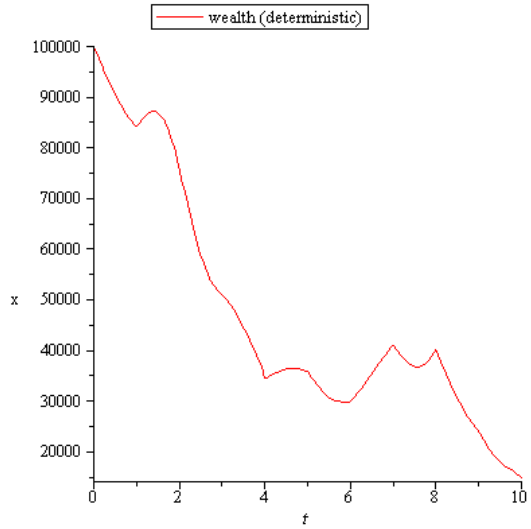


**Figure 3: The optimal expected wealth for 5 samples**

## CONCLUSION

In this work, we have presented a new multi-step method for solving SOC Ps to determine an approximated analytical strategy. Two essential novelty are applied in our approach first, extending of DTM to stochastic type; second, dividing time horizon to some subintervals that this topic cause to added small noises and make one sterategy. The MSDTM is able to reduce noise effectively by breaking the main interval to some small subintervals. The obtained solution is a series in which the coefficients can be computed very accurate. Our presented approach has some advantages than pervious methods such as variational iteration method, Markov chain and etc. First, it provides a continuous curve for solution; second, the convergence of the new approach is guaranteed and the solution is presented as polynomial with suitable approximation. The main its advantage is that we present solution for SOCPs with respect noise system such that are plotted different strategy. This topic especially used in finance and economic problems such as nominal shocks, real shocks and shuck to productivity that, this stuff is aimed at outlining some strategies. At the end, the numerical comparison results are shown the efficiency of this approach on a case study in Merton problem.

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