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Review Article

AN EFFICIENT OPERATIONAL MATRIX METHOD FOR SOLVING TELEGRAPH EQUATIONS IN A LONG TIME PERIOD

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ABSTRACT
In this paper, a set of orthogonal shifted Chebyshev polynomials on $[0, l]$ and a set of orthogonal rational functions on $[0, +\infty)$ are considered. Moreover, a new operational matrix method based on the operational matrices of derivative for the shifted Chebyshev polynomials and rational Chebyshev functions is proposed to solve the second order one dimensional non-homogeneous hyperbolic telegraph equations with initial-boundary conditions on the long time period. In this way, we approximate the solution of the proposed equation with a combination of the shifted Chebyshev polynomials and the rational Chebyshev functions. Some numerical examples are included for demonstrating the efficiency of the method. The results reveal that our method is very effective.

Keywords: *Partial Differential Equations, Telegraph Equation, Shifted Chebyshev Polynomials, Rational Chebyshev Functions*

INTRODUCTION
As we know, many applied problems in science and engineering arise in unbounded domains. In recent years, different spectral methods have been proposed for solving such problems, such as the Hermite and Laguerre spectral methods (Bao and Shen, 2005; Funaro and Kavian, 1991; Guo, 1999). In (Guo *et al.*, 2000), Guo *et al.*, have proposed a method which by mapping a problem under consideration in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials approximates the solutions of the resulting problems. Another approach which usually is used for solving these problems is based on replacing the infinite domain with $[-l, l]$ and the semi-infinite interval with $[0, l]$ by choosing l, sufficiently large. This method is named as the domain truncation (Boyd, 2001). In (Boyd, 1987), Boyd defined new spectral basis functions on the semi-infinite interval, by mapping them to the Chebyshev polynomials, namely rational Chebyshev functions. In (Dehghan and Fakhar-Izadi, 2011), Dehghan and Fakhar-Izadi applied the rational Tau and collocation methods to solve the nonlinear ordinary differential equations on semi-infinite intervals.
A well-known partial differential equation is the telegraph equation. We consider the following form of the this equation:

$$\mathbf{u_{tt}(x,t) + 2\alpha u_t(x,t) + \beta^2 u(x,t) = u_{xx}(x,t) + f(x,t), x \in [0,b], t \in [0,+\infty),} \qquad \mathbf{(1)}$$

which for known real constants α and β is a second order linear hyperbolic telegraph equation in one-dimensional on semi-infinite time. This equation is commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena (El-Azab andEl-Gamel, 2007; Meredith, 1988), and has also been used in modelling the reaction-diffusion processes in various branches of engineering sciences and biological sciences by many researchers, for instance see (Roussy and Pearcy, 1995) and references therein. Moreover it represents a damped wave motion for $\alpha > 0$ and $\beta = 0$. In recent years, much attention has been given in the literature to the development, analysis and implementation of stable methods for the numerical solution of second-order hyperbolic equations, especially, telegraph equation, which is very important in engineering sciences. We advice readers to see (Dehghan, 2005; Gao and Chi, 2007; Mohanty *et al.*, 1996) and references therein.
Approximation by orthogonal families of basic functions has found wide applications in science and engineering. The main advantages of using an orthogonal basis are that the problem under consideration reduces to a system of linear or nonlinear algebraic system of equations. Thus this fact not only simplifies the problem enormously, but also speeds up the computational work during the implementation. This

work can be done by truncating the series expansion in orthogonal basis function for the unknown solution of the problem and in using the operational matrices.

The operational matrix of derivative is given by:

$$\frac{d\Phi(x)}{dx} = D\Phi(x), \tag{2}$$

where $\Phi(x) = [\phi_o(x), \phi_1(x), \dots, \phi_N(x)]^T$, and $\phi_i(x)$ $(i = 0,1, \dots, N)$ are orthogonal basis functions with respect to a specific weight function on a certain interval and D is the operational matrix of derivative of $\Phi(x)$. Notice that D is a constant $(N + 1) \times (N + 1)$ matrix.

The aim of this paper is to use appropriate basis functions for solving the telegraph equation (1) with two types of initial-boundary conditions:

$$\begin{aligned} &u(x,0) = f_0(x), &&u(0,t) = g_0(t),\\ &\lim_{t\to+\infty} u(x,t) = f_1(x), &&u(l,t) = g_1(t), \end{aligned} \tag{3}$$

or

$$\begin{aligned} &u(x,0) = f_0(x), &&u(0,t) = g_0(t),\\ &u_t(x,0) = f_1(x), &&u(l,t) = g_1(t), \end{aligned} \tag{4}$$

where $t \in [0, +\infty)$.

For this purpose, we first introduce two orthogonal bases of functions on the intervals $[0, l]$ and $[0, +\infty)$, generated by the shifted Chebyshev polynomials and rational Chebyshev functions, respectively. Then we describe some properties of the shifted Chebyshev polynomials and obtain a new operational matrix of derivative for these basis functions and also present some useful properties of the rational Chebyshev functions which are used further in this paper. Next a new operational matrix method based on the operational matrices of derivative for the shifted Chebyshev polynomials and rational Chebyshev functions is proposed to solve the above mentioned problems.

This paper is organized as follows: in section 2 we present the shifted Chebyshev polynomials and their properties. In section 3 we introduce the rational Chebyshev functions and also describe some useful properties of these basis functions. In section 4 we propose a new computational method based on the operational matrices of derivative for the shifted Chebyshev polynomials and rational Chebyshev functions. In section 5 the proposed method is applied to several numerical examples.

Chebyshev and Shifted Chebyshev Polynomials

Let $T_n(z)$ be the Chebyshev polynomial of degree n. We recall that $T_n(z)$ is the eigenfunction of the singular Sturm-Liouville problem

$$(1 - z^2)y'' - zy' + n^2y = 0, \; n = 0,1,2,\dots \;, -1 < z < 1. \tag{5}$$

The Chebyshev polynomials are orthogonal with respect to weight function $w(z) = \frac{1}{\sqrt{1-z^2}}$ on $[-1,1]$. The orthogonality of these polynomials is given by:

$$\int_{-1}^{1} T_m(z)T_n(z)w(z)dz = \frac{\pi\gamma_m}{2}\delta_{mn}, \tag{6}$$

where

$$\gamma_m = \begin{cases} 2, & m = 0,\\ 1, & m \ge 1, \end{cases}$$

and δ_{mn} denotes the Kronecker delta.

These basis polynomials can be determined with the recurrence relation (Atkinson and Han, 2009):

$$\begin{aligned} &T_0(z) = 1, \quad T_1(z) = z,\\ &T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \; n \ge 1 \;. \end{aligned} \tag{7}$$

For practical use of these polynomials on the interval of interest $[0, b]$, it is necessary to shift their domain of definition by means of the following substitution (Mason and Handscomb, 2003):

$$z = \frac{2x}{b} - 1, \quad x \in [0, b]. \tag{8}$$

The shifted Chebyshev polynomials $T_n^*(x)$ of degree n on the interval $[0, b]$ are given by:

$$T_n^*(x) = T_n(z) = T_n(\frac{2x}{b} - 1). \tag{9}$$

By using (9) and (7), we may deduce the recurrence relation for $T_n^*(x)$ in the following form:

$$\begin{aligned} &T_0^*(x) = 1, \quad T_1^*(x) = \frac{2x}{b} - 1, \\ &T_{n+1}^*(x) = (\frac{4x}{b} - 2)T_n^*(x) - T_{n-1}^*(x), \; n \geq 1 \; . \end{aligned} \tag{10}$$

The orthogonality condition for these shifted polynomials with respect to the weight function $w_s(x) = \frac{b}{2\sqrt{x(b-x)}}$ on the interval $[0, b]$ is given by:

$$\int_0^b T_m^*(x)T_n^*(x)w_s(x)dx = \frac{b\pi\gamma_m}{4}\delta_{mn}. \tag{11}$$

Any function $f \in L^2_{w_s}[0, b]$, may be expanded by shifted Chebyshev polynomials as:

$$f(x) = \sum_{j=0}^{+\infty} c_j T_j^*(x), \tag{12}$$

where

$$c_j = \frac{(f, T_j^*)_{w_s}}{\|T_j^*\|^2_{w_s}} = \frac{4}{b\pi\gamma_m}\int_0^b f(x)T_j^*(x)w_s(x)dx. \tag{13}$$

If the infinite series in (12) is truncated, then it can be written as:

$$f(x); \sum_{j=0}^{N} c_j T_j^*(x) = C^T\Phi(x), \tag{14}$$

where T denotes transposition and C and $\Phi(x)$ are $N + 1$ column vectors given by:

$$C \triangleq [c_0, c_1, \dots, c_N]^T,$$

$$\Phi(x) \triangleq [T_0^*(x), T_1^*(x), \dots, T_N^*(x)]. \tag{15}$$

Lemma 2.1 Let $T_m^*(x)$ be the Chebyshev polynomials shifted into $[0, b]$. Then we have

$$\frac{d}{dx}T_m^*(x) = \sum_{\substack{k=0 \\ k+m \text{ odd}}}^{m-1} \frac{4m}{b\gamma_k}T_k^*(x). \tag{16}$$

Proof. Suppose that the Chebyshev expansion of function f(z) be as

$$f(z) = \sum_{k=0}^{\infty} \hat{f}_k T_k(z), \tag{17}$$

Then $\frac{d}{dz}f(z)$ can be represented as (Canuto *et al.*, 1988)

$$\frac{d}{dz}f(z) = \sum_{k=0}^{\infty} \hat{f}_k^{(1)} T_k(z), \tag{18}$$

where

$$\hat{f}_k^{(1)} = \frac{2}{\gamma_m}\sum_{\substack{p=k+1 \\ p+k \text{ odd}}}^{m-1} p\hat{f}_p, \tag{19}$$

Now by taking $f(z) = T_m(z)$ in (17), we have $\hat{f}_i = \delta_{im}$, consequently

$$\hat{f}_k^{(1)} = \begin{cases} \frac{2m}{\gamma_k}, & m + k is odd, k \leq m - 1, \\ 0, & otherwise, \end{cases} \tag{20}$$

As a result equation (18) becomes

$$\frac{d}{dz}T_m(z) = \sum_{\substack{k=0 \\ k+m \text{ odd}}}^{m-1} \frac{2m}{\gamma_k}T_k(z), \tag{21}$$

By substituting $z = \frac{2x}{b} - 1$ in Equation (21), we have

$$\frac{d}{dx}T_m^*(x) = \sum_{\substack{k=0 \\ k+m \text{ odd}}}^{m-1} \frac{4m}{b\gamma_k}T_k^*(x). \tag{22}$$

Theorem 2.2 Let $\Phi(x)$ be the shifted Chebyshev polynomials vector defined in (15). Then the derivative of the vector $\Phi(x)$ can be expressed by

$$\frac{d\Phi(x)}{dx} = D_s\Phi(x), \tag{23}$$

where D_s is the $(N + 1) \times (N + 1)$ operational matrix of derivative defined as follows

$$D_{ij}=\begin{cases}\frac{4i}{b\gamma_j} & j=1,2,\cdots,i-1,\text{and}(i+j)\text{odd},\\ 0, & \text{otherwise.}\end{cases} \tag{24}$$

Proof. The ith element of vector $\Phi(x)$ in (15) can be written as

$$\Phi_i(x)=T^*_{i-1}(x) \tag{25}$$

By differentiating this relation and substituting $\frac{d}{dx}T^*_{i-1}(x)$ from relation (16), we have

$$\frac{d\Phi_i(x)}{dx}=\frac{d}{dx}T^*_{i-1}(x)=\sum_{\substack{j=0\\ j+i-1\,\text{odd}}}^{i-2}\frac{4(i-1)}{b\gamma_j}T^*_j(x), \tag{26}$$

So its shifted Chebyshev polynomials expansion has the following form:

$$\frac{d\Phi_i(x)}{dx}=\sum_{\substack{j=1\\ j+i\,\text{odd}}}^{i-1}\frac{4i}{b\gamma_j}\Phi_j(x), \tag{27}$$

Now, by using relations (23) and (27), we obtain:

$$D_{ij}=\begin{cases}\frac{4i}{b\gamma_j} & j=1,2,\cdots,i-1,\text{and}(i+j)\text{odd},\\ 0, & \text{otherwise,}\end{cases} \tag{28}$$

this leads to the desired result.

Rational Chebyshev Functions

The rational Chebyshev functions denoted by $R_n(x)$ are generated from Chebyshev polynomials by employing the mapping $\phi(x)=\frac{x-L}{x+L}$ as:

$$R_n(x)=T_n(\phi(x)) \tag{29}$$

where L is a constant parameter.

The presented mapping for every fixed L maps the semi-infinite interval $[0,\infty)$ into $[-1,1]$. There are some ways for optimizing the positive map parameter L (Guo *et al.*, 2002).

These basis functions can be defined as the following recurrence relations:

$$\begin{aligned}&R_0(x)=1,\\ &R_1(x)=\frac{x-L}{x+L},\\ &R_{n+1}(x)=2\left(\frac{x-L}{x+L}\right)R_n(x)-R_{n-1}(x),\ \ n\geq 1.\end{aligned} \tag{30}$$

Thus, $R_n(x)$ is the nth eigenfunction of the singular Sturm-–Liouville problem (Guo *et al.*, 2002):

$$(x+L)\frac{\sqrt{x}}{L}\frac{d}{dx}\left((x+L)\sqrt{x}\frac{d}{dx}R_n(x)\right)+n^2R_n(x)=0,\quad x\in(0,\infty). \tag{31}$$

Next we present some important properties of the rational Chebyshev functions.

Let $\Lambda=\{x|\ 0<x<\infty\}$. Then $w_r(x)=\frac{\sqrt{L}}{\sqrt{x}(x+L)}$ be a non-negative, integrable, real-valued function on Λ.

We define the Banach space $L^2_{w_r}(\Lambda)$ as follows:

$$L^2_{w_r}=\left\{f:\Lambda\to\mathbb{R}|f\,\text{is measurable and}\ \|f\|_{L^2_{w_r}}<\infty\right\}, \tag{32}$$

where

$$\|f\|_{L^2_{w_r}}=\left(\int_0^\infty|f(x)|^2w_r(x)dx\right)^{\frac{1}{2}}. \tag{33}$$

Here $(.,.)_{w_r}$ denotes the inner product on the $L^2_{w_r}(\Lambda)$.

The orthogonality of the Chebyshev rational functions on the $L^2_{w_r}(\Lambda)$ is given by:

$$(R_m,R_n)_{w_r}=\int_0^\infty R_m(x)R_n(x)w_r(x)dx=\frac{\pi\gamma_m}{2}\delta_{mn}. \tag{34}$$

Any function $f\in L^2_{w_r}(\Lambda)$, may be expanded by Chebyshev rational functions as:

$$f(x)=\sum_{j=0}^{+\infty}c_jR_j(x), \tag{35}$$

where

$$c_j=\frac{(f,R_j)_{w_r}}{\|R_j\|^2_{w_r}}=\frac{2}{\pi\gamma_j}\int_\Lambda f(x)R_j(x)w_r(x)dx. \tag{36}$$

If the infinite series in (35) is truncated, then it can be written as:

$$f(x); \sum_{j=0}^{N} c_j R_j(x) = C^T \Psi(x), \tag{37}$$

where T denotes transposition and C and $\Psi(x)$ are $(N+1)$ column vectors given by:

$$C \triangleq [c_0, c_1, \dots, c_N]^T,$$

$$\Psi(x) \triangleq [R_0(x), R_1(x), \dots, R_N(x)]. \tag{38}$$

The differentiation of the vector $\Psi(x)$, defined in (38) can be expressed as (Parand and Razzaghi, 2004):

$$\frac{d\Psi(x)}{dx} = D_r \Psi(x), \tag{39}$$

where D_r is the $(N+1)\times(N+1)$ operational matrix of differentiation of $\Psi(x)$. It is worth noting that the matrix D_r is a lower-Hessenberg matrix and also can be expressed as $D_r = D_1 + D_2$, where D_1 is a tridiagonal matrix which is given by (Parand and Razzaghi, 2004):

$$D_1 = \text{Tridiag}\left(\frac{7}{4}i, -i, \frac{1}{4}i\right), \quad i = 0,1,\dots,N, \tag{40}$$

and the $(N+1)\times(N+1)$ matrix $D_2 = [d_{ij}]$ is given by:

$$d_{ij} = \begin{cases} 2, & j \geq i-1, \\ k_{ij}\, i\, \xi_j, & j < i-1, \end{cases} \tag{41}$$

where $d_{10} = -1, k_{ij} = (-1)^{i+j+1}, \xi_0 = 1$ and $\xi_j = 2, j > 1$.

Next we investigate the convergence of the series of Chebyshev rational functions.

Consider the space:

$$H^r_{w_r,\mathcal{A}}(\Lambda) = \{f | f \text{isameasurableand} \parallel f \parallel_{r,w_r,\mathcal{A}} < \infty\}, \tag{42}$$

where for non-negative integer r, the norm $\parallel f \parallel_{r,w_r,\mathcal{A}}$ is defined by:

$$\parallel f \parallel_{r,w_r,\mathcal{A}} = \left(\sum_{j=0}^{2r} \left\|(x+1)^{r+j}\frac{d^j}{dx^j} f\right\|^2_{w_r}\right)^{\frac{1}{2}}, \tag{43}$$

and $\mathcal{A}$ is the Sturm-Liouville operator in (31) (Guo et al. 2002),i.e.:

$$[\mathcal{A}f](x) = -\frac{1}{w_r(x)}\frac{d}{dx}\left(\frac{1}{w_r(x)}\frac{d}{dx}f(x)\right). \tag{44}$$

By induction, we have (Guo *et al.*, 2002):

$$[\mathcal{A}^r f](x) = \sum_{j=0}^{2r} (x+1)^{r+j} p_j(x) \frac{d^j}{dx^j} f(x), \tag{45}$$

where $p_j(x)$'s are rational functions which are uniformly bounded on the interval Λ. Therefore, $\mathcal{A}^r$ is a continuous mapping from $H^r_{w_r,\mathcal{A}}(\Lambda)$ to $L^2_{w_r}(\Lambda)$.

Theorem 3.1 (Guo *et al.*, 2002) Suppose $f \in H^r_{w_r,\mathcal{A}}(\Lambda)$, $P_N f(x) = C^T\Psi(x)$ and $r \geq 0$. Then, there is a positive constant $\mathcal{C}$ such that

$$\parallel P_N f - f \parallel_{w_r} \leq \mathcal{C} N^{-r} \parallel f \parallel_{r,w_r,\mathcal{A}} \tag{46}$$

Function Approximation by Shifted Chebyshev Polynomials and Rational Chebyshev Functions

Any sufficiently smooth real function $u(x,t)$ defined on $[0,b]\times[0,+\infty)$ may be expanded in shifted Chebyshev polynomials and Chebyshev rational functions as (Boyd, 2001):

$$u(x,t) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \lambda_{ij} T_i^*(x) R_j(t), \tag{47}$$

where

$$\lambda_{ij} = \frac{\left(\left(u(x,t), T_i^*(x)\right)_{w_s}, R_j(t)\right)_{w_r}}{\|T_i^*\|_{w_s}\|R_j\|_{w_r}}, \tag{48}$$

in which $(.,.)$ denotes the inner product. If the infinite series in Eq.(47) is truncated, then it can be written as

$$u(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} \lambda_{ij} T_i^*(x) R_j(t) = \Phi(x)^T\, U\, \Psi(t), \tag{49}$$

where $\Phi(x)$ and $\Psi(t)$ are $(N+1)$ column vectors defined in (15) and (38) respectively, and U is a $(N+1)\times(N+1)$ known matrix given by:

$$U \triangleq \begin{bmatrix} \lambda_{00} & \cdots & \lambda_{0N} \\ \vdots & \ddots & \vdots \\ \lambda_{N0} & \cdots & \lambda_{NN} \end{bmatrix}. \tag{50}$$

Description of the Proposed Method

In this section, we apply the operational matrices of derivatives of shifted Chebyshev polynomials and rational Chebyshev functions to obtain numerical solutions of telegraph equation in a long time period.

Consider the second order one dimensional linear hyperbolic telegraph equation (1) with the boundary conditions (4). For solving this equation, we first approximate $u(x,t)$ as:

$$u(x,t) = \Phi(x)^T U \Psi(t), \tag{51}$$

where $U = [u_{ij}]_{(N+1)\times(N+1)}$ is an unknown matrix and $\Phi(x)$ and $\Psi(t)$ are the vectors that are defined in (15) and (38), respectively. Now, by employing relations (39) and (23), we can write:

$$\begin{aligned} u_t(x,t) &= \Phi(x)^T\, U\, D_r\, \Psi(t), \\ u_{tt}(x,t) &= \Phi(x)^T\, U\, D_r^2\, \Psi(t), \end{aligned} \tag{52}$$

and

$$u_{xx}(x,t) = \Phi(x)^T\, (D_s^2)^T\, U\, \Psi(t). \tag{53}$$

Also, the function $f(x,t)$ in equation (1) can be approximated as

$$f(x,t) = \Phi(x)^T\, B\, \Psi(t), \tag{54}$$

where $B = [b_{ij}]$ is a $(N+1)\times(N+1)$ known matrix with entries

$$B_{ij} = \left((u(x,t), T_i^*(x))_{w_s}, R_j(t) \right)_{w_r}. \tag{55}$$

Using relations (52)-(55) in equation (1), we get

$$\begin{aligned} &\Phi(x)^T\, U\, D_r^2\, \Psi(t) + 2\alpha\Phi(x)^T\, U\, D_r\, \Psi(t) + \beta^2\Phi(x)^T U\Psi(t) \\ &= \Phi(x)^T\, (D_s^2)^T\, U\, \Psi(t) + \Phi(x)^T\, B\, \Psi(t), \end{aligned} \tag{56}$$

or equivalently:

$$\Phi(x)^T [U\, D_r^2 + 2\alpha U\, D_r + \beta^2 U - (D_s^2)^T\, U - B]\Psi(t) = 0. \tag{57}$$

The entries of $\Phi(x)$ and $\Psi(t)$ are independent, so we get:

$$H = U\, D_r^2 + 2\alpha U\, D_r + \beta^2 U - (D_s^2)^T\, U - B = 0. \tag{58}$$

Relation (58) gives $(N-1)\times(N-1)$ independent equations given by:

$$H_{ij} = 0, \quad i,j = 1,2,\cdots,N-1. \tag{59}$$

We can also approximate the functions $f_0(x), f_1(x), g_0(t)$ and $g_1(t)$ in (4) as:

$$\begin{aligned} f_0(x) &= C_1^T \Phi(x), \qquad & g_0(x) &= C_3^T \Psi(t), \\ f_1(x) &= C_2^T \Phi(x), \qquad & g_1(x) &= C_4^T \Psi(t), \end{aligned} \tag{60}$$

where C_1, C_2, C_3 and C_4 are known vectors of dimension $N+1$. Applying equations (49), (39), (23) and (60) and the initial and boundary conditions (4) we get:

$$\begin{aligned} \Phi(x)^T\, U\, \Psi(0) &= \Phi(x)^T C_1, \qquad & \Phi(0)^T\, U\, \Psi(t) &= C_3^T \Psi(t), \\ \Phi(x)^T\, U\, D_r\, \Psi(0) &= \Phi(x)^T C_2, \qquad & \Phi(l)^T\, U\, \Psi(t) &= C_4^T \Psi(t), \end{aligned} \tag{61}$$

The entries of vectors $\Phi(x)$ and $\Psi(t)$ are independent, so from (61) we obtain:

$$\begin{aligned} U\, \Psi(0) &= C_1, \qquad & \Phi(0)^T\, U &= C_3^T, \\ U\, D_r\, \Psi(0) &= C_2, \qquad & \Phi(l)^T\, U &= C_4^T, \end{aligned} \tag{62}$$

Let

$$\begin{aligned} \Omega_1 &\triangleq U\, \Psi(0) - C_1 = 0, \qquad & \Omega_3 &\triangleq \Phi(0)^T\, U - C_3^T = 0, \\ \Omega_2 &\triangleq U\, D_r\, \Psi(0) - C_2 = 0, \qquad & \Omega_4 &\triangleq \Phi(1)^T\, U - C_4^T = 0. \end{aligned} \tag{63}$$

By choosing the N equations of $\Omega_i = 0$ $(i = 1,2,3,4)$, we get 4N equations, i.e.:

$$\begin{aligned} \Omega_i^j &= 0, \quad j = 0,1,\cdots,N-1, \quad i = 1,2, \\ \Omega_i^j &= 0, \quad j = 1,2,\cdots,N, \qquad i = 3,4, \end{aligned} \tag{64}$$

equations (59) together with (64) give $(N+1)\times(N+1)$ equations, which can be solved for $u_{ij}, i, j = 0,1,\cdots,N$. Thus the approximation of the unknown function $u(x,t)$ can be found.

Numerical Results

In this section, we demonstrate the efficiency of the proposed method by numerical solution of some examples, i.e., two examples for the telegraph equation in the form (1) with the two types of initial-boundary conditions (3) or (4).

Example 1 We consider the hyperbolic telegraph equation (1) with $\alpha = 2, \beta = 1$ and $f(x,t) = -6e^{x-t}$. The initial-boundary conditions are given by:

$$u(x,0) = e^{x}, \qquad u(0,t) = e^{-t},$$
$$\lim_{t\to+\infty} u(x,t) = 0, \qquad u(l,t) = e^{l-t}.$$

The exact solution of this problem is $u(x,t) = e^{x-t}$. In Figures 1, 2, 3 and 4, the space-time graph of the approximate solution with $N = 8$, absolute error between the approximate and exact solutions, approximate and exact solutions for some certain times and absolute error between the approximate and exact solutions for some certain times are presented.

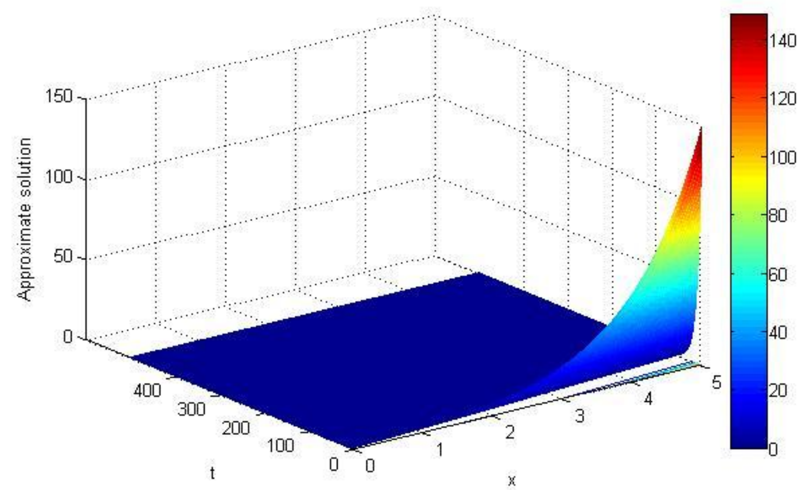

Figure 1: Approximate solution for example 1 with $N = 8$

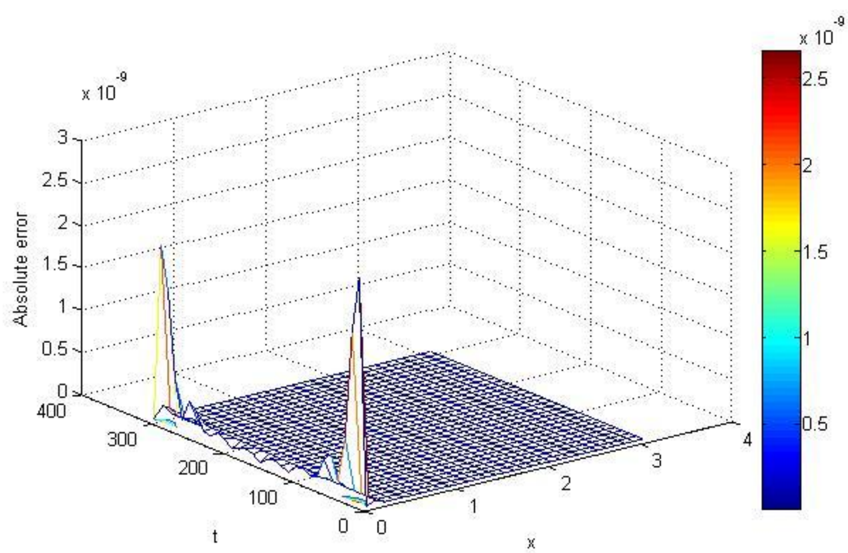

Figure 2: Absolute error for example 1 with $N = 8$

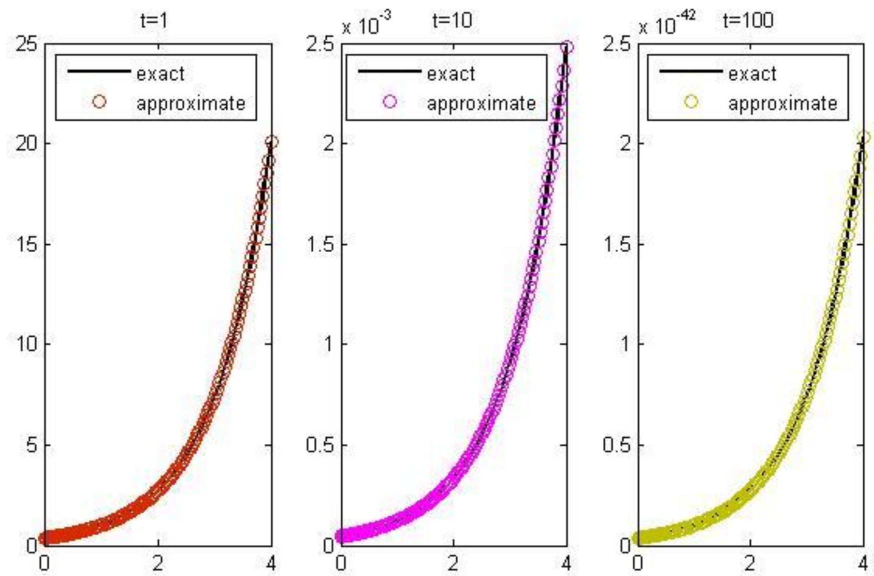

Figure 3: Approximate and exact solutions in certain times for example 1 with N = 8

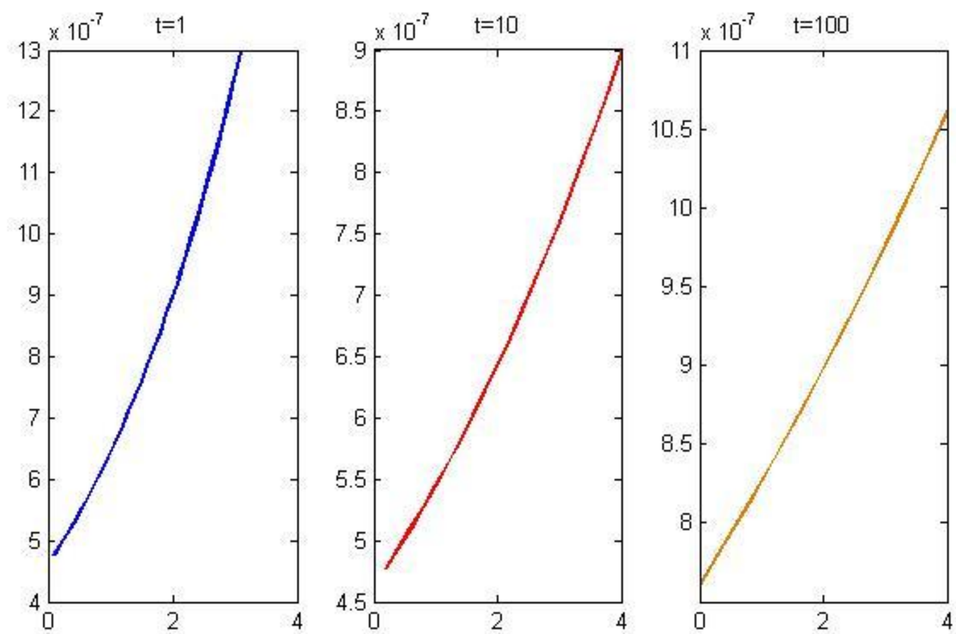

Figure 4: Absolute errors in certain times for example 1 with N = 8

Example 2 We consider the hyperbolic telegraph equation (1) with $\alpha = 2, \beta = 1$ and $f(x,t) = -4\sin(t)\sin(x) + \cos(t)\sin(x)$. The initial boundary conditions are given by:

$$u(x,0) = \sin(x), \qquad u(0,t) = 0,$$
$$u_t(x,0) = 0, \qquad u(l,t) = \cos(t)\sin(l).$$

The exact solution of this problem is $u(x,t) = \cos(t)\sin(x)$. In Figures 5, 6, 7 and 8, the space-time graph of the approximate solution with $N = 8$, absolute error between the approximate and exact solutions, approximate and exact solutions for some certain times and absolute error between the approximate and exact solutions for some certain times are presented.

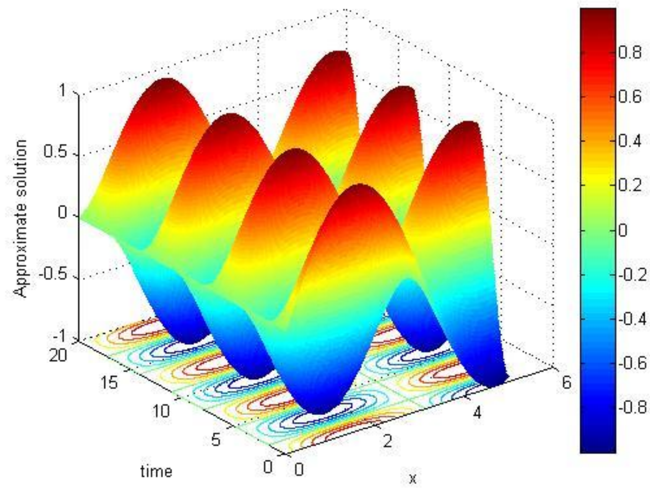

Figure 5: Approximate solution for example 2 with $N = 8$

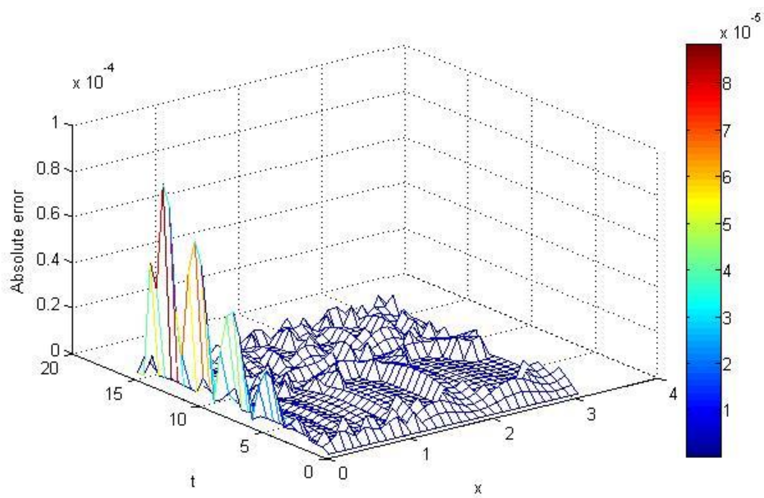

Figure 6: Absolute error for example 2 with $N = 8$

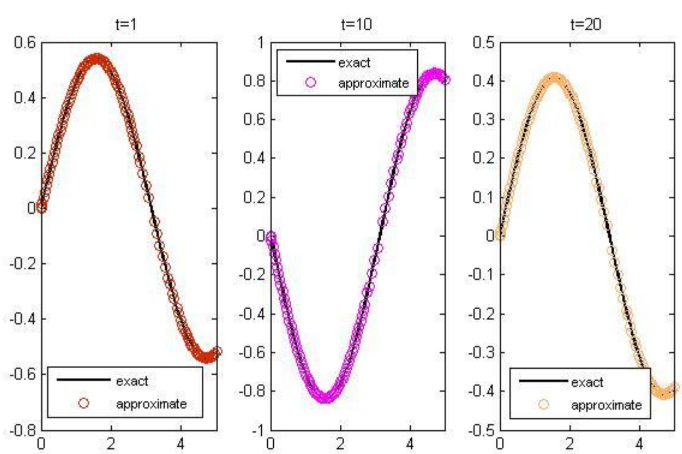

Figure 7: Approximate and exact solutions in certain times for example 2 with $N = 8$

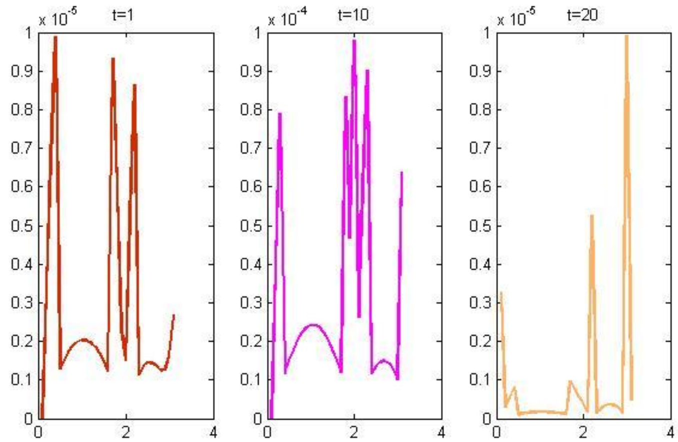

Figure 8: Absolute errors in certain times for example 2 with $N = 8$

CONCLUSION

In the present paper, we proposed the operational matrix method, a very effective and convenient numerical method for approximating the solution of the second order one dimensional non-homogeneous hyperbolic telegraph equations with initial-boundary conditions on the long time period by using the operational matrices of derivative for the shifted Chebyshev polynomials and rational Chebyshev functions. Two examples are presented to demonstrate higher accuracy and simplicity of the proposed method. Also this method can be used for solving other kinds of partial differential equations.

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