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Research Article

LATIN SQUARES AND TRANSVERSAL DESIGNS

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ABSTRACT

We employ a new construction to show that if $g_1 < g_2$ and if there exists a $TD(k, g_1)$ and (k, g_2) , then there exists a $TTD$of type $(g_1 g_2)^k m^1$ for any $0 \leq m \leq \left\lfloor \frac{g_1}{g_2} \right\rfloor g_1^2$. As a corollary, we obtain a new lower bound on the number of mutually orthogonal idempotent Latin squares of side g: $if\ g_1 < g_2$and there exist $r, MOLS$ of side g_1 and $r + 1\ MOLS$ of side g_2, then $N(1^{g_1 g_2}) \geq r$.

***Keywords**: Latin Square, Transversal Design, Pair Wise Balance Design*

INTRODUCTION

In this article we give some explanation about pair wise balance designs and Latin squares. Also we bring some definition about graph and triangle factors and prove there exist sub triangle graphs in complete graph K_{3m} and triangle factors.

Also, we give an upper and lower bound for maximal sets of triangle factors. In addition, show that how to partition blocks of transversal designs to m equivalence classes.

Preliminaries

Let X is a set with v elements. A block of family of sub set with k members is called block design if, every t members subset of distinct elements of X appear exactly λ times in blocks.

A block design shown with $t - (v, k, \lambda)$ and the number of blocks shown with b and the number of point's repetition with r.

Let λ,v be two non-negative integer and K be a set of non-negative integer. A pair wise balance design is pairs (X, B) such that X is the set of points and B is a family of subset of X which is called block and has the following condition:

The number of elements of X is equal to v;

The number of elements of each block is a member of K;

Each subset of distinct elements of X with two elements exactly appears in λ blocks. This pair wise balance design shown with $PBD(v, K, \lambda)$. A block design with parameters $2 - (v, 3,1)$ is called Steiner triple system and shown with $STS(v)$.

Let λ, v be two non-negative integer number and K, M be set of positive integer number. A group divisible design that shown with GDD is a triple (X, G, B).

Such that X is a set of points and $G = \{G_1, G_2, \dots\}$ is a partition of X to groups and B is a family of subset of X with following conditions:

The number of elements of X is equal to v;

The number of elements of each group is a member of M;

The number of elements of each block is a member of K;

Each subset {x,y} such that x,y are in distinct groups exactly appearance in λ blocks;

Each subset {x,y} such that x,y are in one group don't appearance in any blocks.

This group divisible design is shown with $GD(K, \lambda, M, v)$.

A group divisible design is called from type $g_1^{r_1}\ g_2^{r_2} \dots g_s^{r_s}$ if there exists r_i groups with size g_i. Consider $GD(K, M, v)$ of type g^k such that $v = k.g$. This group divisible design is called transversal design and show with $TD(K, g)$.

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An $s \times n^2$ array over n number with s row and n^2column is called orthogonal array, if each row include all number 1 to n and in each column there isn't duplicate pair. This group divisible design is shown with $OA(s,n)$.

Theorem 1.1.18 the following are equivalents:

i) $TD(K,g)$ is exist

ii) $OA(k,g)$ is exist,

iii) There exists k-2 mutually orthogonal Latin square.

Proof.

i)→ $ii)$ Let there exist $TD(K,g)$ and consider groups $TD(K,g)$ as follow

$G_1 = \{1,2,\dots,g\}, G_2 = \{g+1, g+2, \dots, 2g\}, G_k = \{(k-1)g+1, \dots, kg\}$.

We know that $TD(K,g)$ has g^2 blocks. By using above groups make blocks $TD(K,g)$ and column writing this to obtain an $k \times g^2$ array. Now reduce $(j-1)g$ of row j. Then the element of each row are 1 to g. now claim this array is orthogonal array. Consider pair (a,b) in row i and column j . then a is equal to $a + (i-1) \in G_i$ and b is equal to $b + (j-1) \in G_j$ in the first $k \times g^2$ array and since each pair of distinct group appear in a one block, then in both arbitrary row in Column position each pairs are appeared exactly one times.

ii)→ $i)$ let $OA(k,g)$ is exist. We claim that $TD(K,g)$ is exist. For this purpose,

We add $(i-1)g$ to i-th row. Consider the column of new array as the block of transversal design. So the orthogonal array has g^2 column. As regards,

The components of first row are elements of $\{1,2,\dots,g\}$,

The components of second row are elements of $\{g+1, g+2, \dots, 2g\}$,

The components of k-th row are elements of $\{(k-1)g+1, (k-1)g+2, \dots, kg\}$.

So for creation group divisible design it's enough to put the components of i-th row in i-th group. Consider two arbitrary elements a,b of X such that $a \in G_i, b \in G_j$. hence, a is in i-th row and b is in j-th row in the new array. So these are equivalent with $a-(i-1)g, b-(j-1)g$ in the orthogonal array $OA(k,g)$. Then there exist pair $(a-(i-1)g, b-(j-1)g)$ in th column position which is equivalent with (a,b) in one of the blocks of transversal design.

ii)→ $iii)$ let $OA(k,g)$ is exist. We consider this orthogonal array such that the first two rows are as follow:

$1\ 1\ 1 \dots\ 1\ 2\ 2\ 2\ \dots\ 2\ \dots\ g\ g\ g\ \dots\ g$
$1\ 2\ 3\ \dots\ g\ 1\ 2\ 3\ \dots\ g\ \dots\ 1\ 2\ 3\ \dots\ g$

Now we construct Latin square 2-i'th of i-th row of orthogonal array this means that put the first g component in the first row of $g \times g$ array and put the second g component in the second row. This process will continue until the end of the same. Claim that this array is a Latin square. Let there exist the repetitive element t in j-th matrix in r, s-th row. So the pair (j,t) repeated in orthogonal array which is contradiction with definition of orthogonal array. Now we claim that the two Latin square i,j are orthogonal. As regards, when the two squares on the draw there isn't repeated pair, then these two squares are orthogonal. So create $k-2$ mutually orthogonal Latin square of order g.

iii)→ $ii)$ let there exists k-2 mutually orthogonal Latin square of order g. Claim that $OA(k,g)$ is existing. For this purpose, set

$1\ 1\ 1 \dots\ 1\ 2\ 2\ 2\ \dots\ 2\ \dots\ g\ g\ g\ \dots\ g$
$1\ 2\ 3\ \dots\ g\ 1\ 2\ 3\ \dots\ g\ \dots\ 1\ 2\ 3\ \dots\ g$

We put the 1 to $g-th$ rows of Latin square in the $i+2-th$ row of array. So is created the $k \times g^2$ array. We claim that this array is orthogonal array. Consider the first row with the i-th row of array. $j-th$ entries in the first row represents the $j-th$ row of the $j-th$ Latin square and with the fact that all numbers have appeared in this row, therefore, all pairs $(j,1),(j,2),\dots,(j,g)$ appear in a column. Now consider the second and i-th row. Since in j-th column all numbers without repeating appear, therefore, all pairs $(j,1),(j,2),\dots,(j,g)$ appear. Now consider two rows i,j such that $i,j \neq 1,2$. These two lines are represented two Latin square $i-th$ and $j-th$ and are orthogonal. So the two rows have duplicate pairs and all pairs appear.

Structure3.1.1: For make TD(k,g), such that g is a prime number, first we define $X=\{1,2,\dots,k\}\times Z_g$ as points set; it is obvious points of set number are equal k_g. Now For every $1\leq x\leq k$ transversal design groups are defined $G_x=\{x\}\times Z_g$ and we put group's sets in set$G$. Therefore elements of set G partition points set of X. For make blocks, first define set $A=Z_g\times Z_g$, then for every pair $(i,j)\in A$, define blocks b_{ij} as below:

$b_{i\,j} = \{(x, ix + j);\ 1\leq x\leq k\}$.

We must note that $ix+j$ calculated module$g$. Now we put blocks set $b_{i\,j}$ in set B. Set A has g^2 elements, since blocks index $TD(k,g)$ are Elected from set A, therefore number of elements set A is equal to number of elements set B. Since $1\leq x\leq k$, hence size of blocks are equal to k and ordered pairs with distinct first component are not in same block. Now we claim every each binary which elected from two different groups, appear in same block. We consider ordered pair (m,x) of group G_m and (n,y) of group G_n. We must show exist (i,j) in set A such that $(m,x),(n,y)\in b_{ij}$. it is sufficient solve below system for find i,j.

$$\begin{cases} x \overset{g}{\equiv} im+j \\ y \overset{g}{\equiv} in+j \end{cases}$$

Therefore triple (X,G,B) form transversal design $TD(k,g)$.

Lemma 3.1.3: $TD(k,m)$ is Partible if and only if exists $TD(k+1,m)$.

Proof: Let $TD(k,m)$is Partible, we want make $TD(k+1,m)$, Number of elements set component $TD(k,m)$ is km, where number of elements set $TD(k+1,m)$ will be $km+m$. Hence must be added m unit to points set $TD(k,m)$. We add m elements to elements set as a disjoint set $\{a_1,a_2,\dots,a_m\}$. Since $TD(k,m)$is Partible, hence it has m equivalence classes from blocks. We add a_i component to all i'th equivalence classes. Since all elements appear one time in every equivalence class, hence by adding a_i to all blocks of i'th equivalence class, a_i appear by all the points. Since one element has added to all blocks, therefore length of block is equal $k+1$, hence $TD(k+1,m)$ formed.

Inverse: Let exists$TD(k+1,m)$, since $k<k+1$, therefore exists$TD(k,m)$ by Lemma 2.1.3.

We claim $TD(k,m)$ is partible. According Lemma 2.1.3, for make $TD(k,m)$ from$TD(k+1,m)$, must omit all elements one of groups $TD(k+1,m)$ and so omit these elements from blocks, therefore length of blocks reduced one unit.

Let element a_i belong to omitted group, since element a_i, must appear by all elements, hence by omit a_i, union all blocks which contains element a_i, forms an equivalence class from blocks. Therefore by omit m elements from above group, generated transversal design contain $m$equivalence class from blocks, therefore it is partible.

CONSTRUCTION 3.2.2. Let $TD(K, g_1)$ be exists and $TD(K, g_2)$ be a transversal design with s number, g_1 −parallel class of blocks. Then there exist the transversal design $TD(K,\ g_1g_2)$ with sg_1^2 number, parallel class of blocks.

Consider $TD(K, g_2)$ and X as a set number, the set of groups G and the set of blocks B. Let this transversal design has s, g_1 -parallel class of blocks. So $TD(K,\ g_1g_2)$ has one, $(g_2 - sg_1)$ -parallel class of blocks.

Consider $TD(K,\ g_1g_2)$ with set number $X'' = \{(i,h) | 1 \le i \le k\ , h \in \mathbb{Z}_{g1}\}$, groups $G_i'' = \{(i,x) | x \in \mathbb{Z}_{g1}\}$ and the set of blocks B''.

First consider the set of number as $X' = X \times \mathbb{Z}_{g1}$. Considering that, X is a set with kg_2 elements, then X' has kg_1g_2 elements. Also construct $TD(K,\ g_1g_2)$ as $G_i' = G_i \times \mathbb{Z}_{g1}$. We build the blocks as follows:

For each $1 \le j \le s$, put g_1 -parallel class of blocks in a set P_j. So P_j has g_1g_2 blocks. Now for each $x_i \in X$ define a one to one correspondence $\phi_{x_i}(x_1x_2 \dots x_i \dots x_k) \in \mathbb{Z}_{g1}$. We build sets P_j' and B' as follows:

$$P_j' = \left\{\left\{\left(x_1, \phi_{x_1}(b)\right), \left(x_2, \phi_{x_2}(b)\right), \dots \left(x_k, \phi_{x_k}(b)\right)\right\}; b \in P_j, x_i \in G_i\right\},$$

$$B' = \left\{\left\{\left(x_1, \phi_{x_1}(b) + h_1\right), \left(x_2, \phi_{x_2}(b) + h_2\right), \dots \left(x_k, \phi_{x_k}(b)\right)\right\}; b \in P_j, x_i \in G_i\right\}.$$

We repeat this operation for all P_j. So $(sg_1g_2)g_1^2$ block from the $g_1^2g_2^2$ blocks are made such that $(x_i, h), (x_j, h')$ are in a one block if and only if x_i, x_j put in a one block of blocks P. Hence, the number of parallel class of partial transversal design $TD(K,\ g_1g_2)$ are exactly sg_1^2.

To build the rest of the blocks, consider all blocks of $TD(K, g_1)$ and blocks of $TD(K, g_2)$ which applies up $b \notin P_j$. For example consider block $b = \{x_1x_2 \dots x_k\}$

Of $TD(K, g_2)$ and block $b'' = \{(y_1,h_1)(y_2,h_2)\dots(y_k,h_k)\}$ of $TD(K, g_1)$. So $b' = \{(x_1,h_1)(x_2,h_2)\dots(x, h_k)\}$ is a block of $TD(K,\ g_1g_2)$. Then every blocks which is applies up $b \notin P_j$ replace with block g_1^2. So $g_1^2\ (g_2^2 - sg_1g_2)$ block are made. Hence the number of all blocks is

$g_1^2g_2^2 - (sg_1g_2)g_1^2 + (sg_1g_2)g_1^2 = g_1^2g_2^2$.

Therefore $TD(K,\ g_1g_2)$ is a transversal design with sg_1^2 number and parallel class of blocks and one number, $(g_1g_2 - sg_1^2)$ −parallel class.

Example. Let $k = 3, g_1 = 2, g_2 = 3$ and consider $TD(3,3)$ with blocks and groups as follow:

$X_1 = \{1,2,3,4,5,6,7,8,9\}$,

$$G_1^1 = \{1,2,3\}, G_1^2 = \{4,5,6\}, G_1^3 = \{7,8,9\}.$$

$b_1^1 = \{1\ 4\ 7\}, b_1^2 = \{2\ 6\ 8\}, b_1^3 = \{3\ 5\ 9\}$

$b_1^4 = \{1\ 5\ 8\}, b_1^5 = \{2\ 4\ 9\}, b_1^6 = \{3\ 6\ 7\}$

$b_1^7 = \{1\ 6\ 9\}, b_1^8 = \{3\ 4\ 8\}, b_1^9 = \{2\ 5\ 7\}$.

Consider a 2-parallel class of above blocks as follow

$p = \{1\ 4\ 7, 2\ 6\ 8, 3\ 5\ 9, 1\ 5\ 8, 2\ 4\ 9, 3\ 6\ 7\}$.

Now we makw $TD(2,3)$. the set of number and blocks are as follow

$X_2 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$,

$G_2^1 = \{(1,1), (1,2)\}, G_2^2 = \{(2,1), (2,2)\}, G_2^3 = \{(3,1), (3,2)\},$

$b_2^1 = \{(1,1), (2,1), (3,1)\}, b_2^2 = \{(1,1), (2,2), (3,2)\},$

$b_2^3 = \{(1,2), (2,1), (3,2)\}, b_2^4 = \{(1,2), (2,2), (3,1)\},$

Now we construct the blocks of $TD(3,6)$ on the points of $X_3 = X_1 \times \mathbb{Z}_2$ as follow

$X_3 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (4,1), (4,2), (5,1),$
$(5,2), (6,1), (6,2), (7,1), (7,2), (8,1), (8,2), (9,1), (9,2)\}.$

$G_3^1 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

$G_3^2 = \{(4,1), (4,2), (5,1), (5,2), (6,1), (6,2)\}$

$G_3^3 = \{(7,1), (7,2), (8,1), (8,2), (9,1), (9,2)\}.$

Now for every $x \in X$ define a one to one correspondence ϕ_x such that each block consists of x corresponding to the element of $\mathbb{Z}_2$.

$\phi_1(1\ 4\ 7) = 1, \phi_2(2\ 6\ 8) = 1, \phi_3(3\ 5\ 9) = 1, \phi_4(1\ 4\ 7) = 1, \phi_5(3\ 5\ 9) = 1$
,
$\phi_6(2\ 6\ 8) = 1, \quad \phi_7(1\ 4\ 7) = 1, \ \phi_8(2\ 6\ 8) = 1, \ \phi_9(3\ 5\ 9) = 1$

$\phi_1(1\ 5\ 8) = 2, \phi_2(2\ 4\ 9) = 2, \phi_3(3\ 6\ 7) = 2, \phi_4(2\ 4\ 9) = 2, \phi_5(1\ 5\ 8) = 2$

$\phi_6(3\ 6\ 7) = 2, \phi_7(3\ 6\ 7) = 2, \phi_8(1\ 5\ 8) = 2, \phi_9(2\ 4\ 9) = 2.$

And set P' is

$$P' = \left\{\left\{\left(x_1, \phi_{x_1}(b)\right), \left(x_2, \phi_{x_2}(b)\right), \left(x_3, \phi_{x_3}(b)\right)\right\}; b = \{x_1 x_2 x_3\} \in P, x_i \in G_i\right\}.$$

So the elements of P are as follow

$$P' = \{\{(1,1), (4,1), (7,1)\}, \{(2,1), (6,1), (8,1)\}, \{(3,1), (5,1), (9,1)\},$$
$$\{(1,2), (5,2), (8,2)\}, \{(2,2), (4,2), (9,2)\}, \{(3,2), (6,2), (7,2)\},\}.$$

Now we construct the blocks of $TD(3,6)$ as follow

$$b' = \left\{\left\{\left(x_1, \phi_{x_1}(b) + h_1\right), \left(x_2, \phi_{x_2}(b) + h_2\right), \left(x_3, \phi_{x_3}(b) + h_3\right)\right\}; b = \{x_1 x_2 x_3\}, (i, h_i) \epsilon B''\right\} \ \forall b' \in B'$$

Also

$b_3^1 = \{(1,2)(4,2)(7,2)\}, b_3^2 = \{(1,2)(4,1)(7,1)\}, b_3^3 = \{(1,1)(4,2)(7,1)\},$

$b_3^4 = \{(1,1)(4,1)(7,2)\}, b_3^5 = \{(2,2)(6,2)(8,2)\}, b_3^6 = \{(2,2)(6,1)(8,1)\},$

$b_3^7 = \{(2,1)(6,2)(8,1)\}, b_3^8 = \{(2,1)(6,1)(8,2)\}, b_3^9 = \{(3,2)(5,2)(9,2)\},$

$b_3^{10} = \{(3,2)(5,1)(9,1)\}, b_3^{11} = \{(3,1)(5,2)(9,1)\}, b_3^{12} = \{(3,1)(5,1)(9,2)\},$

$b_3^{13} = \{(1,1)(5,1)(8,1)\}, b_3^{14} = \{(1,1)(5,2)(8,2)\}, b_3^{15} = \{(1,2)(5,1)(8,2)\},$

$b_3^{16} = \{(1,2)(5,2)(8,1)\}, b_3^{17} = \{(2,1)(4,1)(9,1)\}, b_3^{18} = \{(2,1)(4,2)(9,2)\},$

$b_3^{19} = \{(2,2)(4,1)(9,2)\}, b_3^{20} = \{(2,2)(4,2)(9,1)\}, b_3^{21} = \{(3,1)(6,1)(7,1)\},$

$b_3^{22} = \{(3,1)(6,2)(7,2)\}, b_3^{23} = \{(3,2)(6,1)(7,2)\}, b_3^{24} = \{(3,2)(6,2)(7,1)\},$

So we have a partial transversal design and the parallel class C_i are as follow

$C_1 = \{\{(1,2)(4,2)(7,2)\}, \{(2,2)(6,2)(8,2)\}, \{(3,2)(5,2)(9,2)\},$
$\{(1,1)(5,1)(8,1)\}, \{(2,1)(4,1)(9,1)\}, \{(3,1)(6,1)(7,1)\}\}$

$C_2 = \{\{(1,2)(4,1)(7,1)\}, \{(2,2)(6,1)(8,1)\}, \{(3,2)(5,1)(9,1)\},$
$\{(1,1)(5,2)(8,2)\}, \{(2,1)(4,2)(9,2)\}, \{(3,1)(6,2)(7,2)\}\}$

$C_3 = \{\{(1,1)(4,2)(7,1)\}, \{(2,1)(6,2)(8,1)\}, \{(3,1)(5,2)(9,1)\},$
$\{(1,2)(5,1)(8,2)\}, \{(2,2)(4,1)(9,2)\}, \{(3,2)(6,1)(7,2)\}\}$

$C_4 = \{\{(1,1)(4,1)(7,2)\}, \{(2,1)(6,1)(8,2)\}, \{(3,1)(5,1)(9,2)\},$
$\{(1,2)(5,2)(8,1)\}, \{(2,2)(4,2)(9,1)\}, \{(3,2)(6,2)(7,1)\}\}.$

Then the 2-parallel class created from the blocks in $TD(3,6)$. the rest of blocks to be created as follow

$b_3^{25} = \{(1,1)(6,1)(9,1)\}, b_3^{26} = \{(1,1)(6,2)(9,2)\}, b_3^{27} = \{(1,2)(6,1)(9,2)\},$

$b_3^{28} = \{(1,2)(6,2)(9,1)\}, b_3^{29} = \{(3,1)(4,1)(8,1)\}, b_3^{30} = \{(3,1)(4,2)(8,2)\},$

$b_3^{31} = \{(3,2)(4,1)(8,2)\}, b_3^{32} = \{(3,2)(4,2)(8,1)\}, b_3^{33} = \{(2,1)(5,1)(7,1)\},$

$b_3^{34} = \{(2,1)(5,2)(7,2)\}, b_3^{35} = \{(2,2)(5,1)(7,2)\}, b_3^{36} = \{(2,2)(5,2)(7,1)\},$

Theorem .3.2.3 let $g_1 < g_2$ and suppose there exists $TD(k, g_1), TD(k+1, g_2)$. then there are TTD of order $(g_1 g_2)^k m^1$ for each $0 \le m \le \left\lfloor \frac{g_1}{g_2} \right\rfloor$.

Proof. Considering that there exists $TD(k+1, g_2)$, by lemma 3.1.3. there exists $TD(k, g_2)$ such that is resolvable and has g_2 parallel class of blocks. Since $g_1 < g_2$, $TD(k, g_2)$ has exactly $s = \left\lfloor \frac{g_1}{g_2} \right\rfloor$, g_1 −parallel classes. Now by construction 3.2.2, . there exists $TD(k, g_1 g_2)$ such that has $s g_1^2$ parallel class of blocks. Now we add a group $\{a_1, a_2, \dots, a_m\}$ to set of groups. Now we add a_1 to all blocks of first class, a_2 to all blocks of second class and similarly a_m to all blocks of $m-th$ class. Thus by 3.2.2.structure, $0 \leq m \leq sg_1^2$. hence we have a $\{k, k+1\} - GDD$ of type $(g_1 g_2)^k m^1$ such that is truncated transversal design.

Theorem .3.3: let $g_1 < g_2$ and there exist r mutually orthogonal latin square of order g_1 and $r+1$ mutually orthogonal latin square of order g_2. Suppose $g_1 \leq \alpha \leq g_1 g_2$, α is a divisor of $g_1 g_2$ and there exist r mutually orthogonal latin square of order α. Then $N(g_1 g_2\ \alpha) \geq r+1$.

Proof. As regards, there exist r mutually orthogonal latin square of order g_1 and $r+1$ mutually orthogonal latin square of order g_2, by theorem 1.1.18 there exist $TD(r+2, g_1), TD(r+3, g_2)$. Also for each α such that $\alpha \leq g_1$, there are $TD(r+2, g_1 g_2)$ that is α −resolvable. According to the assumption there exist r mutually orthogonal Latin square of order α. Hence by theorem 1.1.18. there exist $TD(r+2, \alpha)$. Suppose that $g_1 = \alpha, g_2 = g_1 g_2, s = \frac{g_1 g_2}{\alpha}$. Now by 3.2.2 construction, there are $TD(r+2, g_1 g_2)$ with $\frac{g_1 g_2}{\alpha} \alpha^2 = \alpha g_1 g_2$. Now we add an arbitrary element δ_1 to all the blocks of first parallel classes, an arbitrary element δ_2 to all the blocks of second parallel classes and similarly add an arbitrary element $\delta_{g_1 g_2}$ to all the blocks of $\alpha g_1 g_2$-th parallel classes which is equivalent with $TD(r+3, \alpha g_1 g_2)$. Now by theorem 1.1.18, $N(g_1 g_2\ \alpha) \geq r+1$.

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