SOME NOTIONS BASED UPON STRONGLY GENERALIZED STAR SEMI - CLOSED SETS

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ABSTRACT

The aim of this paper is to define and investigate the concept of strongly generalized star semi – closed sets which is weaker than semi-closed sets (Crossly and Hildebrand, 1971) and stronger than strongly generalized semi-closed sets (El-Maghrabi and Nasef, 2008). Also, some notions in terms of strongly generalized star semi-closed sets are introduced. Further, the concept of strongly semi-star- $T^{\frac{1}{2}}$ space is studied.

Key Words: strongly generalized star semi - closed sets, strongly generalized star semi-open sets , strongly generalized star semi-closure , strongly generalized star semi-interior , strongly generalized star semi-boundary, strongly generalized star semi-exterior and strongly semi-star- $T^{\frac{1}{2}}$ spaces.

INTRODUCTION AND PRELIMINARIES

In 1970, Levine introduced the concept of generalized closed (briefly, g-closed) sets of a topological space. Bhattacharyya and Lahiri (1987) defined and studied the notion of sg-closed sets. In 1990, Arya and Nour introduced the concept of gs- closed sets. Veera Kumar (2001) defined and studied the notion of $g^+$-closed sets. The notion of $g^+$ -closed sets was defined by El-Maghrabi and Nasef (2008).

The purpose of the present paper is to define and investigate the concept of strongly generalized star semi-closed sets and the notion of strongly generalized star -open sets. Moreover, some of their properties are discussed. Further, we define strongly semi-star- $T^{\frac{1}{2}}$ spaces as the space in which every strongly generalized star semi - closed set is semi - closed.

Throughout this paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let X be a space and A be a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$ respectively. A subset A of X is said to be regular-open (Singal and Singal, 1968) (resp. semi – open (Levina, 1963), pre-open (MAshhour et al., 1982), Q-set (Levine, 1961)) if $A = int(cl(A))$ (resp. $A \subseteq cl(int(A))$). The family of all semi – open (resp. semi-closed) sets will be denoted by $SO(X, \tau)$ (resp. $SC(X, \tau)$). The intersection (resp. the union) of all semi- closed (resp. semi-open) sets containing (resp. contained in) A is called the semi closure (resp. the semi-interior) of A and will be denoted by $s - cl(A)$ (resp. $s - int(A)$).

Definition 2.1. A subset of a space $(X, \tau)$ is called:

1- a generalized closed (briefly, g-closed) (Levine, 1970) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

2- a semi generalized-closed (briefly, sg-closed) (Bhattacharyya and Lahiri, 1987) set if $s - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open,
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3- a generalized semi-closed (briefly, gs-closed) (Arya and TM, 1990) set if \( s – cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open,

4- a strongly generalized semi-closed (briefly, \( g^s \)-closed) (El-Maghrabi and Nasef, 2008) set if \( s – cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is g-open,

5- a \( g^* \)-closed (Veera Kumar, 2001) set if \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is g-open.

6- Remark 2.1. The complement of g-closed (resp. sg-closed, gs-closed, \( g^s \)-closed, \( g^* \)-closed) is called g-open (resp. sg-open, gs-open, \( g^s \)-open, \( g^* \)-open).

Definition 2.2. For a subset \( E \) of \((X, \tau)\), we define the following:

(i) \( s – cl_s(E) = \cap \{ A : A \text{ is gs-closed set} \} \) (Dunham, 1982),

(ii) \( s – cl^{*}(E) = \cap \{ A : A \text{ is sg-closed set} \} \) (Sundaram et al., 1991).

Definition 2.3. A topological space \((X, \tau)\) is called:

(i) a \( T_{d} \)-space [6] if every gs-closed set is g-closed,

(ii) a semi- \( T_{\frac{1}{2}} \) space (Bhattactaryya and Lahiri, 1987) if every sg-closed set is semi-closed,

(iii) a \( T_{b} \)-space (Devi, 1994) if every gs-closed set is closed,

(iv) a \( T_{1}^{*} \)-space (Veera Kumar, 2001) if every \( g^* \)-closed set is closed,

(v) a strongly semi- \( T_{\frac{1}{2}} \) (briefly, st. semi- \( T_{\frac{1}{2}} \)) (El-Maghrabi and Nasef, 2008) space if every gs-closed set is \( g^* \)-closed,

(vi) a semi- \( T_{\frac{1}{2}} \) space (El-Maghrabi and Nasef, 2008) if every gs-closed set is semi-closed,

(vii) a semi- \( T_{b} \) space (El-Maghrabi and Nasef, 2008) if every \( g^s \)-closed set is closed,

(viii) a \( T_{\frac{1}{2}} \) space (Levina, 1970) if every g-closed set is closed.

Lemma 2.1 [4]. If \( A \) and \( B \) are two subsets of \( X \), then the following statements are hold:

(i) \( s-cl(A) \) (resp. \( s-int(A) \)) is semi-closed (resp. semi-open),

(ii) \( A \) is semi-closed (resp. semi-open) iff \( A = s – cl(A) \) (resp. \( A = s – int(A) \)),

(iii) \( s – cl(X – A) = X – s – int(A) \) and \( s – int(X – A) = X – s – cl(A) \),

(iv) \( x \in s – cl(A) \) iff for each \( G \in SO(X, \tau) \) containing \( x \), \( G \bigcap A \neq \phi \).

Lemma 2.2 Andrijевич, 1986 Let \( A \) be a subset of a space \( X \). Then, \( s – cl(A) = A \bigcup int(cl(A)) \).

Definition 2.4. A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called:

(i) semi-continuous [10] if \( f^{-1}(U) \) is semi-open in \((X, \tau)\), for every open \( U \) of \((Y, \sigma)\),

(ii) irresolute [5] if \( f^{-1}(U) \) is semi-open in \((X, \tau)\), for every semi-open \( U \) of \((Y, \sigma)\),

(iii) pre- semi-closed [5] if \( f(V) \) is semi-closed in \((Y, \sigma)\), for every semi-closed set \( V \) of \((X, \tau)\).
STRONGLY GENERALIZED STAR SEMI-CLOSED SETS

**Definition 3.1.** A subset $A$ of a space $X$ is called a strongly generalized star semi–closed (briefly, strongly $g^*s$–closed) set if $s-cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is gs–open.

**Remark 3.1.** The family of all strongly $g^*s$–closed subsets of a space $(X, \tau)$ is denoted by $G^*SC(X, \tau)$.

**Remark 3.2.** The concepts of $g$-closed (resp. $g^*$-closed) and strongly $g^*s$–closed sets are independent.

**Example 3.1.** If $X=\{a,b,c,d\}$ with two topologies $\mathcal{T}_1, \mathcal{T}_2$ on $X$ such that: $\mathcal{T}_1 = \{X, \phi, \{a\}, \{a,b\}\}$, $\mathcal{T}_2 = \{X, \phi, \{a\}, \{b,c\}, \{a,b,c\}\}$, then:

1. A subset $A=\{b\}$ of $X$ on $\mathcal{T}_1$ is strongly $g^*s$-closed but not $g$-closed and
2. A subset $B=\{a,b\}$ of $X$ on $\mathcal{T}_1$ is $g^*$-closed but not strongly $g^*s$-closed.

**Example 3.2.** If $X=\{a,b,c,d\}$ with topologies $\mathcal{T}_1, \mathcal{T}_2$ on $X$ such that:

$\mathcal{T}_1 = \{X, \phi, \{c,d\}\}$, $\mathcal{T}_2 = \{X, \phi, \{c\}, \{c,b\}, \{b,c,d\}\}$, then a subset $A=\{a,b,c\}$ of $X$ on $\mathcal{T}_1$ is strongly $g^*s$-closed but not semi–closed. While, a subset $B=\{a\}$ of $X$ on $\mathcal{T}_2$ is $g^*s$-closed but not strongly $g^*s$-closed.

**Example 3.3.** Let $X=\{a,b,c\}$ with topologies $\mathcal{T}_1, \mathcal{T}_2$ on $X$ such that:

$\mathcal{T}_1 = \{X, \phi, \{a,b\}\}$, $\mathcal{T}_2 = \{X, \phi, \{a,\}, \{a,b\}\}$. Then, a subset $C=\{a\}$ of $X$ on $\mathcal{T}_1$ is sg–closed but not strongly $g^*s$–closed. But a subset $D=\{a,c\}$ of $X$ on $\mathcal{T}_2$ is gs–closed but not strongly $g^*s$–closed.
Remark 3.4. The union of two strongly $g^*s$-closed sets need not be strongly $g^*s$-closed. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, the subsets $A = \{a\}$ and $B = \{b\}$ are strongly $g^*s$-closed but their union is not strongly $g^*s$-closed.

Theorem 3.1. A subset $A$ of a space $(X, \tau)$ is strongly $g^*s$-closed if and only if every gs-open set $G$ containing $A$, there exists a semi-closed set $F$ such that $A \subseteq F \subseteq G$.

Proof. Necessity. Let $A$ be a strongly $g^*s$-closed set, $A \subseteq G$ and $G$ be gs-open. Then $s - cl(A) \subseteq G$. Set, $s - cl(A) = F$. Hence, there exists a semi-closed set $F$ such that $A \subseteq F \subseteq G$.

Sufficiency. Assume that $A \subseteq G$ and $G$ is a gs-open set of $X$. Then by hypothesis, there exists a semi-closed set $F$ such that $A \subseteq F \subseteq G$, therefore, $s - cl(A) \subseteq G$. So, $A$ is strongly $g^*s$-closed.

Theorem 3.2. Let $A$ be a strongly $g^*s$-closed set of $X$. Then $(s - cl(A)) - A$ does not contain any nonempty gs-closed set.

Proof. Let $F$ be a gs-closed set such that $F \subseteq (s - cl(A)) - A$. Then $F \subseteq X - A$ this implies that $A \subseteq X - F$. Since, $A$ is strongly $g^*s$-closed and $X - F$ is gs-open, then $s - cl(A) \subseteq X - F$, that is $F \subseteq X - (s - cl(A))$, hence $F \subseteq s - cl(A) \cap (X - (s - cl(A))) = \emptyset$. This shows that $F = \emptyset$.

The converse of the above theorem may not be true as is shown by the following example.

Example 3.4. In Example 3.1, if $A = \{a, b, d\}$ is a subset of $X$ on a topology $\tau_2$, then $(s - cl(A)) - A = \{c\}$ does not contain any nonempty gs-closed set.

Corollary 3.1. Let $A$ be a strongly $g^*s$-closed set of $X$. Then $(s - cl(A)) - A$ does not contain any nonempty gs-closed set.

Proof. Obvious.

Corollary 3.2. Let $A$ be a strongly $g^*s$-closed set. Then $A$ is semi-closed if and only if $(s - cl(A)) - A$ is gs-closed.

Proof. Necessity. Assume that $A$ is strongly $g^*s$-closed and semi-closed sets. Then $s - cl(A) = A$ and hence $(s - cl(A)) - A = \emptyset$ which is gs-closed.

Sufficiency. Suppose that $s - cl(A) - A$ is gs-closed and $A$ is strongly $g^*s$-closed. Then by Corollary 3.1, $(s - cl(A)) - A$ does not contain any nonempty gs-closed subset of $X$. Hence $A$ is semi-closed.

Theorem 3.3. For each $x \in X$, then $\{x\}$ is gs-closed or its complement $X - \{x\}$ is strongly $g^*s$-closed.

Proof. Suppose that $\{x\}$ is not gs-closed. Then its complement is not gs-open. Since, $X$ is the only gs-open set containing $X - \{x\}$, that is, $s - cl(X - \{x\}) \subseteq X$ holds. This implies that $X - \{x\}$ is strongly $g^*s$-closed.

Proposition 3.1. If $A$ is a strongly $g^*s$-closed set and $A \subseteq B \subseteq s - cl(A)$, then $B$ is strongly $g^*s$-closed.

Proof. Let $B \subseteq U$ and $U$ be a gs-open set of $X$. Then $A \subseteq U$. Since, $A$ is strongly $g^*s$-closed, hence $s - cl(A) \subseteq U$, but $B \subseteq s - cl(A)$. Then $s - cl(B) \subseteq U$. Hence, $B$ is strongly $g^*s$-closed.

Proposition 3.2. If $(X, \tau)$ is a topology space and $A \subseteq X$, then $A$ is semi-closed, if one of the following two cases holds:
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(1) If A is strongly $g^*s$-closed and gs-open.
(2) If A is strongly $g^*s$-closed and open.

**Theorem 3.4.** Let A be a subset of a space X, the following are equivalent:
(i) A is regular – open,
(ii) A is open and strongly $g^*s$-closed.

**Proof.** (i) $\implies$ (ii). Let U be a gs-open set containing A and A be a regular-open set. Then, $A \cup \text{int}(cl(A)) \subseteq U$. So, $s - cl(A) \subseteq U$ and therefore A is strongly $g^*s$-closed.

(ii) $\implies$ (i). Since, A is an open and a strongly $g^*s$-closed sets, then by Proposition 3.2(2), A is semi-closed. But, A is pre-open. Therefore, A is regular-open.

**Theorem 3.5.** If A is a subset of a space X, the following are equivalent:
(i) A is clopen,
(ii) A is open, a Q-set and strongly $g^*s$-closed.
(iii) $\implies$ (ii). Since, A is clopen, hence A is both open and a Q-set. Let U be a closed set containing A. Then, $A \cup \text{int}(cl(A)) \subseteq U$. Hence, A is strongly $g^*s$-closed.

**Proof.** (i) $\implies$ (ii). Since, A is clopen, hence A is both open and a Q-set. Let U be a gs-open set containing A. Then, $A \cup \text{int}(cl(A)) \subseteq U$ and so $s - cl(A) \subseteq U$. Hence, A is strongly $g^*s$-closed.

(ii) $\implies$ (i). Hence by Theorem 3.4, A is regular-open. Since, every regular-open set is open, then A is a Q-set, hence A is closed. Therefore, A is clopen.

**STRONGLY GENERALIZED STAR SEMI-OPEN SETS**

**Definition 4.1.** A subset A of a space X is called a strongly generalized star semi-open (briefly, strongly $g^*s$-open) set if $X - A$ is strongly $g^*s$-closed.

**Remark 4.1.** The family of all strongly $g^*s$-open subsets of X is denoted by $G*SO(X, \tau)$.

**Theorem 4.1.** For a subset A of a space X, the following statements are equivalent:
(i) A is strongly $g^*s$-open,
(ii) For each gs-closed set $F \subseteq X$ contained in A, $F \subseteq s - \text{int}(A)$,
(iii) For each gs-closed set $F \subseteq X$ contained in A, there exists a semi-open set $G \subseteq X$ such that $F \subseteq G \subseteq A$.

**Proof.** (i) $\implies$ (ii). Let $F \subseteq A$ and F be a gs-closed set. Then $X - A \subseteq X - F$ which is gs-open. Hence, $s - cl(X - A) \subseteq X - F$. Therefore by Lemma 2.1, (iii), $F \subseteq s - \text{int}(A)$.

(ii) $\implies$ (iii). Let $F \subseteq A$ and F be a gs-closed set. Then by hypothesis, $F \subseteq s - \text{int}(A)$. Set $s - \text{int}(A) = G$, hence $F \subseteq G \subseteq A$.

(iii) $\implies$ (i). Let $X - A \subseteq U$ and U be a gs-open set. Then $X - U \subseteq A$ and by hypothesis, there exists a semi-open set G such that $X - U \subseteq G \subseteq A$, that is, $X - A \subseteq X - G \subseteq U$. Therefore, by Theorem 3.1, $X - A$ is strongly $g^*s$-closed. Hence, A is strongly $g^*s$-open.

**Remark 4.2.** Every semi-open set is strongly $g^*s$-open but, the converse is not true as is shown by the following example.

**Example 4.1.** In Example 3.2, a subset $A = \{d\}$ of X on $\tau_1$ is strongly $g^*s$-open but not semi-open.

**Remark 4.3.** The intersection of two strongly $g^*s$-open sets need not be strongly $g^*s$-open, as is illustrated by the following example.
Example 4.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, the subsets $A = \{b, c, d\}$ and $B = \{a, c, d\}$ are strongly $g^*s$-open but their intersection is not strongly $g^*s$-open.

Theorem 4.2. If $A$ is a strongly $g^*s$-open subset of $X$, then $U = X$, whenever $U$ is gs- open and $s - \text{int}(A) \cup (X - A) \subseteq U$.

Proof. Suppose that $U$ is a gs- open set and $s - \text{int}(A) \cup (X - A) \subseteq U$. Then, $X - U \subseteq (X - (s - \text{int}(A))) \cap A$ and by Lemma 2.1(iii), $X - U \subseteq s - \text{cl}(X - A) - (X - A)$. Hence, by Corollary 3.1, $X - U = \emptyset$ which implies that $X = U$.

Proposition 4.1. If $s - \text{int}(A) \subseteq B \subseteq A$ and $A$ is strongly $g^*s$-open, then $B$ is strongly $g^*s$-open.

Proof. Since, $s - \text{int}(A) \subseteq B \subseteq A$, then $X - A \subseteq X - B \subseteq X - s - \text{int}(A)$, hence by Lemma 2.1(iii), $X - A \subseteq X - B \subseteq s - \text{cl}(X - A)$, then by Theorem 3.4, $X - B$ is strongly $g^*s$-closed. Therefore, $B$ is strongly $g^*s$-open.

Lemma 4.1. Let $A \subseteq X$ be a strongly $g^*s$-closed set. Then $s - \text{cl}(A) - A$ is strongly $g^*s$-open.

Proof. Let $F$ be a gs- closed set such that $F \subseteq (s - \text{cl}(A) - A)$. Since $A$ is strongly $g^*s$-closed, then by Corollary 3.1, $F = \emptyset$. Therefore, $\emptyset \subseteq s - \text{int}(s - \text{cl}(A) - A)$. Hence, by Theorem 4.1, $s - \text{cl}(A) - A$ is strongly $g^*s$-open.

STRONGLY GENERALIZED STAR SEMI-TOPOLOGICAL OPERATIONS

Definition 5.1. In a space $(X, \tau)$, if $A \subseteq X$, then the strongly generalized star semi-closure (briefly, strongly $g^*s - \text{cl}(A)$) and the strongly generalized star semi-interior (briefly, strongly $g^*s - \text{int}(A)$) of $A$ are defined by respectively.

- $\text{strongly } g^*s - \text{cl}(A) = \bigcap \{F_i : A \subseteq F_i, F_i \in \text{strongly } G^* \text{SC}(X, \tau)\}$,
- $\text{strongly } g^*s - \text{int}(A) = \bigcup \{G_i : G_i \subseteq A, G_i \in \text{strongly } G^* \text{SO}(X, \tau)\}$.

According to the above definition, it is easy to see that $A \subseteq \text{strongly } g^*s - \text{cl}(A)$ and $\text{strongly } g^*s - \text{int}(A) \subseteq A$.

Proposition 5.1. For a topological space $(X, \tau)$, then:

(i) If $A \subseteq F$ and $F$ is strongly $g^*s$-closed, then $A \subseteq \text{strongly } g^*s - \text{cl}(A) \subseteq F$.
(ii) If $G \subseteq A$ and $G$ is strongly $g^*s$-open, then $G \subseteq \text{strongly } g^*s - \text{int}(A) \subseteq A$.

Proof. Obvious.

Proposition 5.2. For a space $(X, \tau)$, if A is a subset of X, then the following statements are hold:

(i) If $A$ is strongly $g^*s$-closed, then $A = \text{strongly } g^*s - \text{cl}(A)$.
(ii) If $A$ is strongly $g^*s$-open, then $A = \text{strongly } g^*s - \text{int}(A)$.

Proof. Obvious.

Lemma 5.1. For a space $(X, \tau)$, if A is a subset of X, then

(i) $\text{strongly } g^*s - \text{cl}(X - A) = X - \text{strongly } g^*s - \text{int}(A)$,
(ii) $\text{strongly } g^*s - \text{int}(X - A) = X - \text{strongly } g^*s - \text{cl}(A)$.

Proof. Obvious from Definition 5.1.
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**Lemma 5.2.** If \( A \) is a subset of a space \((X, \tau)\), then \( x \in strongly \, g^*s - \text{int}(A) \) if and only if there exists a strongly \( g^*s \)-open set \( W \) such that \( x \in W \subseteq A \).

**Proof.** The necessity. Let \( x \in strongly \, g^*s - \text{int}(A) \). Then, \( x \in \bigcup_{i=1}^{n} W_i \) is strongly \( g^*s \)-open, \( W_i \subseteq A \).

Therefore there exists at least \( W \) contains \( x \) such that \( x \in W \subseteq \bigcup_{i=1}^{n} W_i = strongly \, g^*s - \text{int}(A) \subseteq A \). Hence, \( x \in W \subseteq A \).

The sufficiency. Assume that there exists a strongly \( g^*s \)-open set \( W \) such that \( x \in W \subseteq A \). Then, \( X - A \subseteq X - W \). By Lemma 5.1, \( x \in X - strongly \, g^*s - \text{cl}(X - A) \). Hence, by Lemma 5.1, \( x \in strongly \, g^*s - \text{int}(A) \).

**Lemma 5.3.** If \( A \) is a subset of a space \((X, \tau)\), then \( x \in strongly \, g^*s - \text{cl}(A) \) if and only if for each \( G \in strongly \, G^*SO(X, \tau) \) containing \( x \), \( G \cap A \neq \phi \).

**Proof.** The necessity. Let \( x \in strongly \, g^*s - \text{cl}(A) \). Then, \( x \in X - strongly \, g^*s - \text{cl}(A) \), hence by Lemma 5.1, \( x \in strongly \, g^*s - \text{int}(X - A) \). Then, by Lemma 5.2, there exists a strongly \( g^*s \)-open set \( G \) such that \( x \in G \subseteq X - A \). So, \( A \cap G = \phi \).

The sufficiency. Assume that there exists a strongly \( g^*s \)-open set \( G \) containing \( x \) such that \( A \cap G = \phi \). Then, \( A \subseteq X - G \) which is a strongly \( g^*s \)-closed set. Therefore, by Proposition 5.1, \( A \subseteq strongly \, g^*s - \text{cl}(A) \subseteq X - G \), but \( x \notin X - G \). Hence, \( x \notin strongly \, g^*s - \text{cl}(A) \).

**Remark 5.1.** For a space \((X, \tau)\), if \( A \subseteq X \), we have:

\[
\text{int}(A) \subseteq s\text{-int}(A) \subseteq strongly \, g^*s - \text{int}(A) \subseteq A \subseteq s - cl_*(A) \subseteq s - cl^*(A) \subseteq strongly \, g^*s - \text{cl}(A) \subseteq s - cl(A).
\]

The converse of the above remark is not true as is shown by (Crossley and Hildebrand, 1971; Maki et al., 1996) and the following example.

**Example 5.1.** Let \( X = \{a, b, c, d\} \) with topologies \( \tau_1 = \{X, \emptyset, \{c, d\}\} \), \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \).

(i) If \( A = \{b, c\} \) of \( X \) on \( \tau_1 \), then \( s - cl(A) = X \) but strongly \( g^*s - cl(A) = \{a, b, c\} \). Therefore, \( s - cl(A) \subset strongly \, g^*s - cl(A) \).

(ii) Also, if \( B = \{b\} \) of \( X \) on \( \tau_2 \), then \( s - cl^*(B) = B \) but strongly \( g^*s - cl(B) = \{b, c\} \). Therefore, \( strongly \, g^*s - cl(B) \subset s - cl^*(B) \).

(iii) Further, if \( C = \{c\} \) of \( X \) on \( \tau_1 \), then \( s - int(C) = \emptyset \) but strongly \( g^*s - int(C) = C \). Therefore, \( strongly \, g^*s - int(C) \subset s - int(C) \).

**Definition 5.2.** Let \((X, \tau)\) be a space and \( A \subseteq X \). Then, the strongly generalized star semi-boundary of \( A \) (briefly, \( strongly \, g^*s - b(A) \)) is defined by

\[
\text{strongly} \, g^*s - b(A) = strongly \, g^*s - \text{cl}(A) \cap strongly \, g^*s - \text{cl}(X - A).
\]

**Theorem 5.1.** In a space \((X, \tau)\), if \( A \) and \( B \) are two subsets of \( X \), then the following statements are hold:

(i) \( strongly \, g^*s - b(A) = strongly \, g^*s - b(X - A) \),
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(ii) strongly $g^s - b(A) = strongly g^s - cl(A) - strongly g^s - int(A)$,
(iii) strongly $g^s - b(A) \cap strongly g^s - int(A) = \phi$,
(iv) strongly $g^s - b(A) \cup strongly g^s - int(A) = strongly g^s - cl(A)$.

Definition 5.3. If $(X, \tau)$ is a space and $A \subseteq X$, then the set $X - strongly g^s - cl(A)$ is called the strongly generalized star semi-exterior of $A$ and is denoted by $strongly g^s - ext(A)$.

Each point $P \in X$ is called a strongly $g^s$-exterior point of $A$, if it is strongly $g^s$-interior point of $X - A$.

Theorem 5.2. For a space $(X, \tau)$ and $A$ be a subset of $X$, the following statements are hold:
(i) $strongly g^s - ext(A) = strongly g^s - ext(X - A)$,
(ii) $strongly g^s - ext(A) \cap strongly g^s - b(A) = \phi$,
(iii) $strongly g^s - ext(A) \cup strongly g^s - b(A) = strongly g^s - cl(X - A)$.

Proof. (i) By Definition 5.3, $strongly g^s - ext(A) = X - strongly g^s - cl(A)$, hence by Lemma 5.1, $strongly g^s - ext(A) = strongly g^s - int(X - A)$.
(ii) Obvious. Since, $strongly g^s - ext(A) \cup strongly g^s - b(A) = strongly g^s - int(X - A) \cup strongly g^s - b(X - A) = strongly g^s - cl(X - A)$.

STRONGLY SEMI STAR $T_{1/2}$ SPACES

Definition 6.1. A topological space $X$ is said to be:
(1) strongly semi-star-$T_{1/2}$ (briefly, $st.semi^* - T_{1/2}$) if every strongly $g^s$-closed set in $X$ is semi-closed,
(2) strongly semi-star-$T_p$ (briefly, $st.semi^* - T_p$) if every strongly $g^s$-closed set in $X$ is closed.

Example 6.1. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a, b, \{a, c\}\}$, Then $(X, \tau_1)$ is $st.semi^* - T_{1/2}$ and $(X, \tau_2)$ is $st.semi^* - T_p$.

Lemma 6.1. For a space $(X, \tau)$, the following are hold:
(1) Every $st.semi^* - T_p$ (resp. semi-$T_p$, semi-$T_b$, $T_b$) space is $st.semi^* - T_{1/2}$.
(2) Every $T_b$-space is $st.semi^* - T_p$.

Proof. (1) Let $A$ be a strongly $g^s$-closed subset of $X$. Since, $(X, \tau)$ is $st.semi^* - T_p$, then $A$ is closed. Therefore, $A$ is semi-closed. Hence $(X, \tau)$ is $st.semi^* - T_{1/2}$.
(2) Let $A$ be a strongly $g^s$-closed subset of $X$. Then, $A$ is $gs$-closed. Since, $(X, \tau)$ is $T_b$, then $A$ is closed. Hence, $(X, \tau)$ is $st.semi^* - T_p$.

The converses of the above theorem need not be true as may be seen by the following example.

Example 6.2. Let $X = \{a, b, c\}$ with the following topologies:
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(1) \( \tau_2 = \{X, \phi \{a\}, \{a, b\}\} \),

(2) \( \tau_3 = \{X, \phi \{a, b\}, \{c\}\} \).

Then, from Example 6.1, \((X, \tau_1)\) is a \( st. semi^{*} - T_{\frac{1}{2}} \) space but it is not \( st. semi^{*} - T_{\frac{1}{2}}p \) (resp. semi- \( T_{\frac{1}{2}}p \), \( T_b \)).

Further, \((X, \tau_2)\) is \( st. semi^{*} - T_{\frac{1}{2}} \) but it is not semi- \( T_{\frac{1}{2}}b \). Also, \((X, \tau_3)\) is a \( st. semi^{*} - T_{\frac{1}{2}}p \) space but it is not \( T_{\frac{1}{2}}b \).

Remark 6.1. (1) \( st. semi - T_{\frac{1}{2}} \) and \( st. semi^{*} - T_{\frac{1}{2}}p \) spaces are independent,

(2) \( semi - T_{\frac{1}{2}} \) and \( st. semi^{*} - T_{\frac{1}{2}}p \) spaces are independent,

(3) \( st. semi^{*} - T_{\frac{1}{2}} \) and \( T_d \) spaces are independent,

(4) \( st. semi - T_{\frac{1}{2}} \) and \( st. semi^{*} - T_{\frac{1}{2}} \) spaces are independent.

Example 6.3 Let \( X = \{a, b, c\} \) with the following topologies:

(1) \( \tau_4 = \{X, \phi \{a\}, \{b\}, \{a, b\}\} \),

(2) \( \tau_5 = \{X, \phi \{a, b\}\} \). Then (1) From Example 6.2, \((X, \tau_3)\) is a \( st. semi^{*} - T_{\frac{1}{2}}p \) space but it is not \( st. semi - T_{\frac{1}{2}} \) and \((X, \tau_2)\) is \( st. semi^{*} - T_{\frac{1}{2}} \) but it is not \( st. semi^{*} - T_{\frac{1}{2}}p \).

(2) Also, from Example 6.2, \((X, \tau_3)\) is \( st. semi^{*} - T_{\frac{1}{2}}p \) but it is not semi- \( T_{\frac{1}{2}} \) and \((X, \tau_4)\) is semi- \( T_{\frac{1}{2}} \) but it is not \( st. semi^{*} - T_{\frac{1}{2}}p \).

(3) Further, from Example 6.2, \((X, \tau_2)\) is \( st. semi^{*} - T_{\frac{1}{2}} \) but it is not \( T_d \) and \((X, \tau_5)\) is \( T_d \) but it is not \( st. semi^{*} - T_{\frac{1}{2}} \).

(4) Furthermore, from Example 6.2, \((X, \tau_3)\) is \( st. semi^{*} - T_{\frac{1}{2}} \) but it is not semi- \( T_{\frac{1}{2}} \) and \((X, \tau_5)\) is \( st. semi^{*} - T_{\frac{1}{2}} \).

Remark 6.2. We can summarize the following diagram by using (Devi, 1994; El-Maghrabi and Nasef, 2008; Levine, 1970; Veera Kumar, 2001) and the above results.
Theorem 6.1. For a topological space \((X, \tau)\), the following statements are equivalent:

(i) \((X, \tau)\) is \(stsemi^* - T_{\frac{1}{2}}\).

(ii) Every singleton of \(X\) is either gs-closed or semi-open.

Proof. (i)\(\Rightarrow\)(ii). Let \(x \in X\) and \(\{x\}\) be not gs-closed. Then by Theorem 3.3, \(X - \{x\}\) is strongly \(g^*s\)-closed. Hence by hypothesis, \(X - \{x\}\) is semi-closed. Therefore, \(\{x\}\) is semi-open.

(ii)\(\Rightarrow\)(i). Let \(A \subseteq X\) be a strongly \(g^*s\)-closed set and \(x \in s - cl(A)\), we need to show that \(x \in A\). For consider the following two cases:

case (1). The set \(\{x\}\) is gs-closed. Then, if \(x \not\in A\), there exists a gs-closed set in \((s - cl(A)) - A\). So, by Corollary 3.1, \(x \in A\).

case (2). The set \(\{x\}\) is semi-open. Since, \(x \in s - cl(A)\), then \(\{x\} \cap A \neq \emptyset\). Thus, \(x \in A\). Hence, in both cases, \(x \in A\). This show that \(s - cl(A) \subseteq A\). So, \(A\) is semi-closed.

Theorem 6.2. For a topological space \((X, \tau)\), the following statements are equivalent:

(i) \((X, \tau)\) is \(stsemi^* - T_{\frac{1}{2}}\).

(ii) Every residual singleton of \(X\) is gs-closed.

(iii) \(X\) is \(stsemi^* - T_{\frac{1}{2}}\).

Proof. (i) \(\Rightarrow\) (ii). By Theorem 6.1, every singleton set of \(X\) is semi-open or gs-closed and since, a non-void residual set can not be semi-open at the same time, then every residual singleton set of \(X\) is gs-closed.

(ii) \(\Rightarrow\) (i). Since, every singleton set is either semi-open or residual, then by hypothesis, every singleton set of \(X\) is either semi-open or gs-closed. Hence by Theorem 6.1, \(X\) is \(stsemi^* - T_{\frac{1}{2}}\).

Theorem 6.3. A space \((X, \tau)\) is \(stsemi^* - T_{\frac{1}{2}}\) if and only if \(SO(X, \tau) = strongly\ \ G^*SO(X, \tau)\).  

Proof. Necessity. Let \((X, \tau)\) be a \(stsemi^* - T_{\frac{1}{2}}\) space and \(A \in strongly\ \ G^*SO(X, \tau)\). Then, \(X - A\) is strongly \(g^*s\)-closed. Hence by hypothesis, \(X - A\) is semi-closed. Hence, \(A \in SO(X, \tau)\).
Therefore, strongly \( G^*SO(X, \tau) \subseteq SO(X, \tau) \). Also, By Remark 4.2, \( SO(X, \tau) = strongly \ G^*SO(X, \tau) \).

Sufficiency. Let \( SO(X, \tau) = strongly \ G^*SO(X, \tau) \) and \( A \) be a strongly \( g^*s \)-closed set. Then \( X - A \in strongly \ G^*SO(X, \tau) \), hence by hypothesis, \( X - A \in SO(X, \tau) \). Therefore, \( A \) is semi-closed.

**Corollary 6.1.** A space \((X, \tau)\) is \( st.sem\i^* - T_p \) if and only if \( \tau = strongly \ G^*SO(X, \tau) \).

**Definition 6.2.** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be:

1. strongly generalized star semi -continuous (briefly, strongly \( g^*s \)-continuous) if \( f^{-1}(V) \) is strongly \( g^*s \)-closed in \((X, \tau)\), for every closed set \( V \) of \((Y, \sigma)\).
2. strongly generalized star semi-irresolute (briefly, strongly \( g^*s \)-irresolute) if \( f^{-1}(V) \) is strongly \( g^*s \)-closed in \((X, \tau)\), for every strongly \( g^*s \)-closed \( V \) of \((Y, \sigma)\).

**Theorem 6.4.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be strongly \( g^*s \)-continuous. Then, \( f \) is semi -continuous, if \((X, \tau)\) is \( st.sem\i^* - T_{1/2} \).

**Proof.** Let \( V \subseteq Y \) be a closed set. Then \( f^{-1}(V) \) is strongly \( g^*s \)-closed in \( X \). But \( (X, \tau) \) is \( st.sem\i^* - T_{1/2} \), hence, \( f^{-1}(V) \) is semi - closed. Therefore, \( f \) is semi-continuous.

**Theorem 6.5.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be strongly \( g^*s \)-irresolute. Then, \( f \) is irresolute, if \((X, \tau)\) is \( st.sem\i^* - T_{1/2} \).

**Proof.** Obvious.

**Theorem 6.6.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be onto, pre semi - closed and strongly \( g^*s \)-irresolute mappings. Then \((Y, \sigma)\) is \( st.sem\i^* - T_{1/2} \), if \((X, \tau)\) is \( st.sem\i^* - T_{1/2} \).

**Proof.** Let \( V \subseteq Y \) be a strongly \( g^*s \)-closed set. Then \( f^{-1}(V) \) is strongly \( g^*s \)-closed in \( X \). But \((X, \tau)\) is \( st.sem\i^* - T_{1/2} \), hence \( f^{-1}(V) \) is semi-closed. Therefore \( f(f^{-1}(V)) = V \) is semi - closed in \( Y \).

**Corollary 6.2.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be onto, closed and strongly \( g^*s \)-irresolute mappings. Then \((Y, \sigma)\) is \( st.sem\i^* - T_p \), if \((X, \tau)\) is \( st.sem\i^* - T_p \).

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