ON SLIGHTLY P-CONTINUOUS MULTIFUNCTIONS

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ABSTRACT

The concepts of p-continuity between topological spaces are introduced by Thangavelu and Rao. The purpose of this paper is to introduce upper and lower slightly p-continuous multifunctions and investigate their relationships with other multifunctions.

INTRODUCTION AND PRELIMINARIES

In this chapter the concepts of slightly p-continuous and slightly q-continuous multifunctions are introduced and their properties are investigated.

Definitions

Definition 1.1:
Let A be a subset of a topological space X, then A is called
• a p-set[2] if $cl(int(A)) \subseteq int(cl(A))$
• α-open[1] if $A \subseteq int(cl(int(A))$ and α-closed if $cl(int(cl(A))) \subseteq A$

Definition 1.2:
A multifunction $F: X \rightarrow (Y, \sigma)$ is said to be upper p-continuous on X if for every open set $x \in X$, for every $V$ in $Y$ containing $F(x)$ there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \subseteq V$.

Definition 1.3:
A multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ is said to be lower p-continuous on X if for every $x \in X$ if for every open set $V$ in $Y$ with $V \cap F(x) \neq \emptyset$ there is a p-set $U$ such that $x \in U$ and $F(U) \cap V \neq \emptyset$ for every $u \in U$.

Definition 1.4:
A multifunction $F: X \rightarrow Y$ is said to be weakly injective if for any two subsets $A_1$ and $A_2$ of $X$, $F(A_1) \cap F(A_2) \neq \emptyset$ implies $A_1 \cap A_2 \neq \emptyset$. A multifunction $F: X \rightarrow Y$ is said to be weakly bijective if it is both weakly injective and weakly surjective.

Lemma 1.5:
Let a multifunction $F: X \rightarrow Y$ is weakly surjective. Then for any subset $A$ of $X$ $Y \setminus F(A) \subseteq F(X \setminus A)$.

Proof: Let $y \in Y \setminus F(A)$. Then $y \in Y$ and $y \notin F(x)$ for every $x \in A$. Since $F$ is weakly surjective there is a subset $B$ of $X$ such that $F(B) = Y$ that implies $y \in F(b)$ for some $b \in B$. If $b \in A$ then $F(b) \subseteq F(A)$ so that $y \in F(A)$ contradicting $y \in Y \setminus F(A)$. Therefore $b$ cannot lie in $A$. Thus $b \in B \setminus A$. This shows that $Y \setminus F(A) \subseteq F(B \setminus A) \subseteq F(X \setminus A)$.

Lemma 1.6:
Let a multifunction $F: X \rightarrow Y$ is weakly injective. Then for any subset $A$ of $X$ $Y \setminus F(A) \supseteq F(X \setminus A)$. Moreover if $F: X \rightarrow Y$ is weakly bijective then for any subset $A$ of $X$, $Y \setminus F(A) = F(X \setminus A)$.

Proof: Let $y \in F(X \setminus A)$. Then $y \in F(x)$ for some $x \in X \setminus A$. Clearly $y \in Y$. Suppose $y \notin F(A)$. Then $y \notin F(a)$ for some $a \in A$. Therefore $F(x) \cap F(a) \neq \emptyset$. Since $F$ is weakly injective, by Definition 1.4 $\{x\} \cap \{a\} \neq \emptyset$ that implies $x = a$. This is absurd as $x \notin A$ and $a \in A$. This proves $Y \setminus F(A) \supseteq F(X \setminus A)$. Since $F$ is weakly bijective the above arguments together with Lemma 1.5 imply that $Y \setminus F(A) = F(X \setminus A)$.

The following four Lemmas will be useful in sequel.

Lemma 1.7:
A subset $B$ of $X$ is a p-set if and only if $X \setminus B$ is also a p-set. [2]
Lemma 1.8:
A subset $B$ of $X$ is a p-set in $(X,\tau)$ if and only if it is p-set in $(X,\tau^a)$ where $\tau^a$ is a topology of $\alpha$-open sets in $(X,\tau)$. [2]

Lemma 1.9:
The intersection of a p-set with a clopen set is again a p-set [2].

Lemma 1.10:
Let $Y$ be a clopen set in $X$ and $B \subseteq Y \subseteq X$. Then $B$ is a p-set in $X$ if and only if it is a p-set in $Y$. [2]

Upper Slightly $p$-Continuity

In this section upper slightly p-continuous functions are introduced and characterized

Definition 2.1:
A multifunction $F: (X,\tau) \to (Y,\sigma)$ is said to be upper slightly p-continuous on $X$ if for every $V \in CO(Y,\sigma)$ containing $F(x)$ there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \subseteq V$. An upper slightly p-continuous multifunction is upper slightly $m$-continuous if $m_X = p(X,\tau)$.

Theorem 2.3:
A multifunction $F: (X,\tau) \to (Y,\sigma)$ is upper slightly p-continuous on $X$ if and only if for every $x$ and for every clopen set $B$ in $Y$ with $B \cap F(x) = \emptyset$, there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \subseteq V$. An upper slightly p-continuous multifunction is upper slightly $m$-continuous if $m_X = p(X,\tau)$.

Proof:
Suppose $F: (X,\tau) \to (Y,\sigma)$ is upper slightly p-continuous on $X$. Let $x \in X$ and $B$ be a clopen set in $Y$ such that $B \cap F(x) = \emptyset$. Then $Y \setminus B$ is also clopen in $Y$ with $F(x) \subseteq Y \setminus B$. Using Definition 2.1 there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \subseteq V$. Thus $F(U) \setminus B = \emptyset$. Conversely, we assume that for every $x$ and for every clopen set $B$ in $Y$ with $B \cap F(x) = \emptyset$, there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \setminus B = \emptyset$. Fix $x \in X$ and a clopen set $V$ in $Y$ with $F(x) \subseteq V$. Then $F(x) \cap (Y \setminus V) = \emptyset$. Then by the assumptions there is a p-set $U$ with $x \in U$ and $F(U) \cap (Y \setminus V) = \emptyset$ that implies $F(U) \subseteq V$. By using Definition 2.1 $F$ is upper slightly p-continuous.

Theorem 2.4:
Suppose $F: (X,\tau) \to (Y,\sigma)$ is weakly bijective. Then $F$ is upper slightly p-continuous on $X$ if and only if for every $x$ and for every clopen set $B$ in $Y$ with $B \cap F(x) = \emptyset$, there is a p-set $W$ in $X$ such that $x \in W$ and $B \subseteq F(W)$.

Proof:
Suppose $F$ is upper slightly p-continuous. Fix $x \in X$. Let $B$ be a clopen set in $Y$ and $B \cap F(x) = \emptyset$. Since $F$ is upper p-continuous by using Theorem 2.3 there is a p-set $U$ with $x \in U$ such that $F(U) \subseteq B = \emptyset$. This implies that $F(U) \subseteq Y \setminus B$ that implies $Y \setminus F(U) \subseteq B$. Since $F$ is weakly bijective, $F(X \cup U) = Y \setminus F(U) \subseteq B$. Taking $W = X \cup U$ and using Lemma 1.7 we see that $W$ is a p-set with $x \in W$ and $B \subseteq F(W)$. Conversely, let us assume that for every $x$ and for every clopen set $B$ in $Y$ with $B \cap F(x) = \emptyset$, there is a p-set $W$ in $X$ such that $x \in W$ and $B \subseteq F(W)$. In order to prove that $F$ is upper slightly p-continuous fix $x \in X$ and a clopen set $B$ with $B \cap F(x) = \emptyset$. Then by our assumption there is a p-set $W$ in $X$ such that $x \in W$ and $B \subseteq F(W)$. Therefore $B \cap (Y \setminus F(W)) = \emptyset$. Since $F$ is weakly bijective, by Lemma 1.6, $Y \setminus F(W) = F(X \setminus W)$ that implies $B \cap F(X \setminus W) = \emptyset$. Then by applying Theorem 2.3, $F$ is upper slightly p-continuous.

Theorem 2.5:
A multifunction $F: (X,\tau) \to (Y,\sigma)$ is upper slightly p-continuous on $X$ if and only if $F: (X,\tau^a) \to (Y,\sigma)$ is upper p-continuous.

Proof:
Let $F : (X,\tau)\to (Y,\sigma)$ be upper p-continuous. Suppose $x \in X$ and $V$ is a clopen set in $Y$ with $F(x) \subseteq V$. Then by using Definition 2.1, there is a p-set $U$ in $(X,\tau)$ such that $F(U) \subseteq V$. Again by using Lemma 1.8, $U$ is also a p-set in $(X,\tau^a)$. This proves that $F: (X,\tau^a) \to (Y,\sigma)$ is upper slightly p-continuous. Conversely we assume that $F: (X,\tau^a) \to (Y,\sigma)$ is upper slightly p-continuous. Let $x \in X$ and $V$ be an open set in $Y$ with $F(x) \subseteq V$. Then by using Definition 1.3, there is a p-set $U$ in $(X,\tau^a)$ such that $F(U) \subseteq V$. Again by using Lemma 1.8, $U$ is also a p-set in $(X,\tau)$. This proves that $F: (X,\tau) \to (Y,\sigma)$ is upper p-continuous. Thus (i) am proved.
The next two Lemmas on p-sets are used to show that the restriction of an upper p-continuous multifunction to a clopen set is upper p-continuous.

**Theorem 2.6:**
If a multifunction $F: X \rightarrow Y$ is upper slightly p-continuous and $X_0$ is clopen, then the restriction $F |_{X_0} : X_0 \rightarrow Y$ is upper p-continuous.

**Proof:** Suppose a multifunction $F: X \rightarrow Y$ is upper slightly p-continuous on $X$. Fix $x \in X_0$ and $V$ a clopen set in $Y$. Since $F$ is upper slightly p-continuous on $X$, by **Definition 2.1**, there is a p-set $U$ in $X$ such that $F(U) \subseteq V$. Let $U_0 = U \cap X_0$. Then using **Lemma 1.9**, $U_0$ is a p-set in $X$. Again by using **Lemma 1.10**, $U_0$ is a p-set in $X_0$. Since $F(U_0) \subseteq F(U) \subseteq V$, by **Definition 2.1**, $F |_{X_0}$ is upper p-continuous at $x$ and hence on $X_0$.

**Theorem 2.9:**
A multifunction $F: X \rightarrow Y$ is upper slightly p-continuous on $X$ if and only if for every $x$ in $X$ and for every clopen set $V$ in $Y$ with $x \in F^+(V)$, there is a p-set $U$ in $X$ such that $x \in U$ and $U \subseteq F^+(V)$.

**Proof:** We assume that $F$ is upper slightly p-continuous on $X$. Let $x \in X$. Let $V$ be a clopen set in $Y$ such that $x \in F^+(V)$. This implies $F(x) \subseteq V$. Then by **Definition 2.1**, there is a p-set $U$ in $X$ such that $x \in U$ and $F(U) \subseteq V$.

If $u \in U$, then $u \in F^+(V)$, that implies $F(u) \subseteq V$. Therefore $\bigcup_{u \in U} F(u) \subseteq V$ and hence $F(U) \subseteq V$.

Therefore by **Definition 2.1**, $F$ is upper p-continuous at $x$ and hence on $X$.

**Theorem 2.10:**
Let $X$ be a space in which the family of all p-sets is closed under arbitrary union. A multifunction $F: X \rightarrow Y$ is upper slightly p-continuous on $X$ if and only if for every clopen set $V$ in $Y$, $F^+(V)$ is a p-set in $Y$.

**Proof:** Suppose $F$ is upper slightly p-continuous on $X$. Let $V$ be a non-empty clopen set in $Y$. Let $x \in F^+(V)$. Then by **Theorem 2.9**, there is a p-set $U$ in $X$ such that $x \in U$ and $U \subseteq F^+(V)$. Therefore $F^+(V)$ is a union of p-sets. By hypothesis, $F^+(V)$ is a p-set. Conversely, we assume that for every clopen set $V$ in $Y$, $F^+(V)$ is a p-set in $X$. Let $V$ be a clopen set with $x \in F^+(V)$. Since $F^+(V)$ is a p-set in $X$, taking $U = F^+(V)$ and using **Theorem 2.9**, $F$ is upper slightly p-continuous.

**Example 2.11:**
Let $X = Y = \{a, b, c, d\}$, and $\tau = \sigma = \{\emptyset, \{b, c, d\}, \{a, b\}, \{a, c\}, \{d\}, \{b, a\}, \{c, d\}, \{b\}, \{X\}, \{X\}\}$. Define $F: (X, \tau) \rightarrow (X, \sigma)$ by $F(a) = \{a, d\}$, $F(b) = \{a, c\}$, $F(c) = \{c, d\}$, and $F(d) = \{a, c\}$. $F$ is upper slightly p-continuous.

**Lower Slightly p-Continuity**
In this section lower slightly p-continuous functions are introduced and characterized.

**Definition 3.1:**
A multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ is said to be lower slightly p-continuous on $X$ if for every clopen set $V$ in $Y$ with $V \cap F(x) \neq \emptyset$ there is a p-set $U$ such that $x \in U$ and $F(u) \cap V \neq \emptyset$ for every $u \in U$.

**Theorem 3.2:**
Let a multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ be weakly injective. If $F$ is lower slightly p-continuous on $X$ then for every $x$ and for every clopen set $B$ in $Y$ with $B \cap F(x) \neq \emptyset$, there is a p-set $U$ in $X$ such that $x \in U$ and $B \cup F(X \setminus U) \neq Y$ for every $x \in U$.

**Proof:** Suppose $F$ is lowering slightly p-continuous. Fix $x$ in $X$ and a clopen subset $B$ of $Y$ with $B \cap F(x) \neq \emptyset$. Then $F(x) \subseteq Y \setminus B$. Since $Y \setminus B$ is clopen in $Y$ and since $(Y \setminus B) \cap F(x) = F(x) \neq \emptyset$, by using **Definition 3.1**, there is a p-set $U$ in $X$ with $x \in U$ and $F(u) \cap V \neq \emptyset$ for every $u \in U$. This implies $(Y \setminus F(u)) \cup (Y \setminus V) \neq Y$.
that implies \((Y \cap F(u)) \cup B \neq Y\). Since \(F\) is weakly injective by Lemma 3.2.8, \(F(X \setminus u) \subseteq Y \setminus F(u)\), that implies \(F(X \setminus u) \cup B \neq Y\) for every \(u \in U\).

**Theorem 3.3:**
A multifunction \(F: (X, \tau) \to (Y, \sigma)\) is lower slightly \(p\)-continuous on \(X\) if and only if \(F: (X, \tau^0) \to (Y, \sigma)\) is lower slightly \(p\)-continuous;

**Proof:** Let \(F: (X, \tau) \to (Y, \sigma)\) be lower slightly \(p\)-continuous. Suppose \(x \in X\) and \(V\) is a clopen set in \(Y\) with \(F(x) \cap V \neq \emptyset\). Then by using Definition 3.1, there is a \(p\)-set \(U\) in \((X, \tau)\) such that \(x \in U\) and \(F(u) \cap V \neq \emptyset\) for all \(u \in U\). Again by using Lemma 1.9, \(U\) is also a \(p\)-set in \((X, \tau^0)\). This proves that \(F: (X, \tau^0) \to (Y, \sigma)\) is lower slightly \(p\)-continuous. Conversely we assume that \(F: (X, \tau^0) \to (Y, \sigma)\) is lower slightly \(p\)-continuous. Let \(x \in X\) and \(V\) be a clopen set in \(Y\) with \(F(x) \cap V \neq \emptyset\). Then by using Definition 3.1, there is a \(p\)-set \(U\) in \((X, \tau^0)\) such that \(F(u) \cap V \neq \emptyset\) for all \(u \in U\). Again by using Lemma 1.9, \(U\) is also a \(p\)-set in \((X, \tau)\). This proves that \(F: (X, \tau) \to (Y, \sigma)\) is lower slightly \(p\)-continuous.

**Theorem 3.4:**
If a multifunction \(F: X \to Y\) is lower slightly \(p\)-continuous and \(X_0\) is clopen, then the restriction \(F/_{X_0}: X_0 \to Y\) is lower slightly \(p\)-continuous.

**Proof:** Suppose a multifunction \(F: X \to Y\) is lower slightly \(p\)-continuous on \(X\). Fix \(x \in X_0\) and \(V\) is a clopen set in \(Y\) with \(F(x) \cap V \neq \emptyset\). Since \(F\) is lower slightly \(p\)-continuous on \(X\), by Definition 3.1, there is a \(p\)-set \(U\) in \(X\) such that \(x \in U\) and \(F(u) \cap V \neq \emptyset\) for all \(u \in U\). Let \(U_0 = U \cap X_0\). Then using Lemma 1.9, \(U_0\) is a \(p\)-set in \(X_0\). Again by using Lemma 1.10, \(U_0\) is a \(p\)-set in \(X_0\). Since \(U_0 \subseteq U\), it follows that \(F(u) \cap V \neq \emptyset\) for all \(u \in U_0\). So by Definition 3.1, \(F/_{X_0}\) is lower slightly \(p\)-continuous on \(X\) and hence on \(X_0\).

**Theorem 3.7:**
A multifunction \(F: X \to Y\) is lower slightly \(p\)-continuous on \(X\) if and only if for every \(x \in X\) and for every clopen set \(V\) in \(Y\) with \(x \in F^{-1}(V)\) there is a \(p\)-set \(U\) in \(X\) such that \(x \in U\) and \(U \subseteq F^{-1}(V)\).

**Proof:** We assume that \(F\) is lowering slightly \(p\)-continuous on \(X\). Let \(x \in X\). Let \(V\) be a clopen set in \(Y\) such that \(x \in F^{-1}(V)\). This implies \(F(x) \cap V \neq \emptyset\). Then by Definition 3.1, there is a \(p\)-set \(U\) in \(X\) such that \(x \in U\) and \(F(u) \cap V \neq \emptyset\) for all \(u \in U\). Now \(u \in U \Rightarrow F(u) \cap V \neq \emptyset \Rightarrow u \in F^{-1}(V)\). This proves that \(U \subseteq F^{-1}(V)\). Conversely we assume that for every \(x \in X\) and for every clopen set \(V\) in \(Y\) with \(x \in F^{-1}(V)\) there is a \(p\)-set \(U\) in \(X\) such that \(x \in U\) and \(U \subseteq F^{-1}(V)\). Now let \(x \in X\) and let \(V\) be a clopen set in \(Y\) with \(F(x) \cap V \neq \emptyset\). Since \(F(x) \cap V \neq \emptyset\), \(x \in F^{-1}(V)\). By our assumption there exists a \(p\)-set \(U\) of \(X\) such that \(x \in U \subseteq F^{-1}(V)\). If \(u \in U\), then \(u \in F^{-1}(V)\) that implies \(F(u) \cap V \neq \emptyset\) for all \(u \in U\). Therefore by Definition 3.1, \(F\) is lowering \(p\)-continuous on \(X\).

Examples can be constructed to show that the reverse implications are not true.

**Example 3.8:**
Let \(X = \{a, b, c, d\}\) and \(\tau = \sigma = \{\emptyset, \{b, c, d\}, \{a, b\}, \{c, d\}, \{b\}, X\}\). Define \(F: (X, \tau) \to (X, \sigma)\) by \(F(a) = \{a, d\}\), \(F(b) = \{a, c\}\), \(F(c) = \{c, d\}\) and \(F(d) = \{a, b\}\). \(F\) is upper slightly \(p\)-continuous.

**REFERENCES**