ABSTRACT
In this paper, I explain cumulative sum (CUSUM) control scheme in brief, present numerical example and it is verified that the CUSUM is an efficient alternative to Shewhart procedures. It is shown that CUSUM scheme is more efficient in detecting small shifts in the mean of a process. The analysis of ARL for CUSUM control chart shows better performance than Shewhart control chart when it is desired to detect the shifts in the mean of size 1.5 sigma or less. In the present paper, simulation is carried out to calculate the ARL values using C-programs. Observing these values it is seen that approximately the same values of ARL are obtained by simulation method which were obtained by analytical studies.

Key Words: Average Run Length, CUSUM Control Chart, EWMA Control Chart, Simulation, Statistical Process Control

INTRODUCTION
The control chart is the widely applied technique for controlling the industrial processes. The pioneer work on Statistical Process Control (SPC) was done by Walter A. Shewhart in the earlier 1920’s. Shewhart developed the basis for the control chart and the state of statistical control. Shewhart’s chart are effective for detecting large changes in process parameters; however Shewhart chart may take a long time to detect a small persistent shift in the process parameter. The ability to detect smaller parameter shifts can be improved by using a chart based on a statistic that corporate information from past samples in addition to current samples. One such chart is Cumulative Sum (cusum) control chart developed by Page (1954).This chart plots the cumulative sums of deviations of the sample values of a quality characteristic from a target value against time. It is noted that Shewhart’s control chart for mean is very effective if the magnitude of the shift is 1.5 -sigma or larger ( Montgomery 2001). Some authors namely Duncan (1974), Lucas (1976), Hawkins (1981), Lucas and Saccucci (1982a, 1990) stated that the cusum control chart is much more efficient than the usual control chart for detecting smaller variations in the average. There are two ways to represent a cusum, the tabular or algorithmic cusum and the V- mask cusum. Of these two forms the tabular form of the cusum is practiced more. So we consider the construction and use of the tabular cusum in this paper. An excellent discussion of cusum control scheme is given by Hawkins and Olwell (1998). The rest of the paper is organized as follows. Section 2 gives a review on the cusum control chart with principles of cusum control scheme. In Section 3, I gave an example in which Shewhart and cusum charts are plotted and compared. In section 4, shift detection properties of the Cusum are verified by evaluating the average run length (ARL) values. Finally section 5 concludes the paper by a brief summary of results and discussion.

Cumulative sum (CUSUM) control chart
Cusum may be constructed for individual observations as well as rational subgroups. First the case of individual observations is considered.

The Tabular CUSUM for Monitoring the Process Mean: Let $x_i$ be the $i^{th}$ observation on the process. $x_i$ has normal distribution with mean $\mu_0$ and standard deviation $\sigma$ (known or estimable ), when the process is under control. Sometimes, $\mu_0$ is taken to be the target value for the quality characteristic X.

The tabular cusum works by accumulating the deviations from $\mu_0$ that are above target with one statistic $C^+$ and that are below target with another statistic $C^-$. The statistics $C^+$ and $C^-$ are called one sided upper and lower cusum respectively. They are calculated as

CUMULATIVE SUM CONTROL CHART
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\[ C_i^+ = \max \{ 0, \ x_i - (\mu_0 + k) + C_{i-1}^+ \} \]
\[ C_i^- = \max \{ 0, \ (\mu_0 - k) - x_i + C_{i-1}^- \} \quad ---- (1.1) \]

with the starting value \( C_0^+ = C_0^- = 0 \). The value of \( k \) is called the reference or allowable value and it is often chosen about halfway between the target \( \mu_0 \) and the shift of mean which one is interested in detecting. Thus,

\[ k = \frac{1}{2} \cdot |\mu_1 - \mu_0| \]

The CUSUM values \( C_i^+ \) and \( C_i^- \) accumulate deviation from the target value \( \mu_0 \) that are greater than \( k \). If either of the two exceeds the decision interval \( H \), the process is said to be out of control. A reasonable value of \( H \) is five times the process standard deviation \( \sigma \). Here \( H = h * \sigma \) and \( K = k * \sigma \) are the parameters of the CUSUM chart (Montgomery 2001).

**Rational Subgroups:** In case of rational subgroups (of sizes \( n>1 \)), replacing \( x_i \) by \( \bar{x}_i \) the sample mean and \( \sigma \) with \( \frac{\sigma}{\sqrt{n}} \) in the above formulas, the tabular CUSUM is obtained.

The cusum control scheme consists of collecting \( k \) samples, each of size \( n \) and mean of each sample are computed. Then, the cumulative sums of deviations of the sample mean of a quality characteristic, from a target value are calculated. These cumulative sums are then plotted against the time (sample number). For example, suppose the samples of size \( n \geq 1 \) are collected and \( \bar{x}_j \) is the mean of the \( j^{th} \) sample. If \( \mu_0 \) is the target for the process mean, then the cusum control chart obtained by plotting the quantity, \( C_i = \sum_{j=1}^{i} (\bar{x}_j - \mu_0) \) against the \( i^{th} \) subgroup. \( C_i \) is called the cumulative sum up to the \( i^{th} \) subgroup.

**Example**

To illustrate the cusum control scheme an example is given, where set of simulated observations \( X_i \)'s \( (X_i \)'s are i.i.d. with common variance \( \sigma^2 \) ) is taken from a normal process. The target value is zero. For the first ten observations, the process is shifted upward by 0.5 standard deviation. For the next ten observations, the process mean was shifted upward by one standard deviation and for the next ten it is shifted to 1.5 standard deviation. For this scheme, the parameters are chosen as \( H = 5 \) and \( K = 0.5 \). This gives in control ARL is equal to 500. The values of two Cusum control statistic \( C_i^+ \) and \( C_i^- \) are computed using the equation (1.1).

From the chart, it is seen that the values of the cusum control statistic lies around the target value for the first ten observations but then increases after the shift of 1 \( \sigma \) in the process mean takes place and gives out of control signal at 21\(^{st} \) observation (Figure1). For the same data the Exponentially Weighted Moving Average control (EWMA) chart gives out of control signal at 26\(^{th} \) observation while Shewhart chart does not give out of control signal up to the shift of 1.5 \( \sigma \) (Figure 2).

**The Average Run Length of CUSUM Control Chart**

The average run length (ARL), for certain level of quality is the average number of the samples taken before the out of control signal. For CUSUM chart, the parameters \( K \) and \( H \) together with the sample size \( n \) are selected to obtain appropriate ARL's at acceptable quality level say \( \mu_0 \) and rejectable quality level say \( \mu_1 \). One would like to see a high ARL, ARL0 (at \( \mu_0 \) ) when the process is in control and a low ARL, ARL1 (at \( \mu_1 \) ) when the process mean has shifted to an unsatisfactory level.
Figure 1: Cusum control scheme. One sided upper cusum $C_i^+$ verses sample number are plotted as blue color points and lower cusum $C_i^-$ are plotted as pink color points. The parameter values are $k=0.5$ and $H=5$. This control scheme (for individual observations) gives an out of control signal at 22$^{nd}$ observation and thus is very effective for small shifts.

Figure 2: Shewhart control scheme with usual $3 \sigma$ control limits. This control scheme does not give an out of control signal for the first 30 observations, indicating that the Shewhart control scheme is slow for detecting smaller shifts.

There have been many analytical studies of CUSUM ARL performance. According to these studies, there are some general recommendations for selecting $H$ and $K$. Let $H = h*\sigma$ and $K = k*\sigma$, where $\sigma$ is the process standard deviation. Using $h = 4$ or 5 and $k = \frac{1}{2}$, one can get good ARL properties against shift of about 1 standard deviation in the process mean. To illustrate the usefulness of these recommendations, the two sided ARL $s$ of the tabular cusum and the ARL $s$ of the Shewhart chart are given in Table 1 (Montgomery 2001). The simulated values are given in brackets.

**ARL of Cusum control chart by simulation:**

The following algorithm is used in our simulation study to estimate ARL values of CUSUM for given $H$, $k$, $d$ (shift).

Step 1: Input $N$, $n$.
Step 2: Set range for $d$, $d = 0, (0.25/\sqrt{n}), 2$. 

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*Note: The diagrams and tables are not included in the text format. The figures are not presented here.*
Step 3 : Counter I = 0 , SRL = 0.
Step 4 : I = I+1 , RL = 0.
Step 5 : If I > N , then go to step 16.
Step 6 : Sum = 0.
Step 7 : Z = 0.
Step 8 : Generate 12 random samples from U (0,1 ).
Step 9 : Compute SNV : Z ← ∑ Ui-6.
Step 10 : Z=Z+d and sum = sum +Z.
Step 11 : Repeat the steps 7 to 10 , n times.
Step 12 : Compute x ← sum /n.
Step 13 : C_0^+ = 0.
Step 14 : Compute C_i^+ = max [ 0, y_i- k + C_{i-1}^+]
Step 15 : If C > H go to step 16 otherwise process is in control and RL ← RL+1 and go to step 6.
Step 16 : SRL ← SRL+RL.
Step 17 : Go to step 4.
Step 18 : Compute ARL ← SRL/ N.
Step 19 : Print d and ARL.
Step 20 : Go to step 2 for the next value of d.
Step 21 : End.

The ARL values of two sided cusum are obtained by using the equation

\[
\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}.
\]

By symmetry, \(ARL_0^+ = ARL_0^-\).

When h=4, the in control ARL for two sided cusum is given by,

\[
\frac{1}{ARL_0^+} = \frac{1}{ARL_0^+} + \frac{1}{ARL_0^-} = \frac{1}{339.11} + \frac{1}{339.11} = \frac{2}{339.11} = \frac{1}{169.5576}.
\]

Then \(ARL_0 = 169.5576\). (by simulation).

This is very close to the true ARL value of 168 as shown in Table 1

Similarly, for h=5, \(ARL_0 = 471.9207\). (by simulation).

The exact value shown in Table 1 is 465.

The out of control ARL for two sided cusum is given by,

\[
\frac{1}{ARL_i^+} = \frac{1}{ARL_i^+} + \frac{1}{ARL_i^-} = \frac{1}{ARL_i^+}. \quad (since \quad ARL_i^- \quad is \quad large)
\]

If the mean shifts by 2-σ, then ARL_1 of two sided cusum by simulation, for h=4 is 3.35 and for h=5 is 4.01. The corresponding values in the table are 3.34 and 4.01 respectively which are very close to the simulated values.
Table 1: ARL performance of the tabular cusum with \( k = \frac{1}{2} \) and \( h = 4 \) and \( 5 \).

<table>
<thead>
<tr>
<th>Shift in mean (multiple of ( \sigma ))</th>
<th>( h = 4 )</th>
<th>( h = 5 )</th>
<th>Shewhart ( \bar{X} ) control chart</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>168(169.55)</td>
<td>465(471.9)</td>
<td>371 (374.1)</td>
</tr>
<tr>
<td>0.25</td>
<td>74.2(77.55)</td>
<td>139(142.0)</td>
<td>281.14(281.59)</td>
</tr>
<tr>
<td>0.50</td>
<td>26.6(26.6)</td>
<td>38.0(37.86)</td>
<td>155.22(158.18)</td>
</tr>
<tr>
<td>0.75</td>
<td>13.3(13.28)</td>
<td>17.0(17.0)</td>
<td>81.22(82.35)</td>
</tr>
<tr>
<td>1.00</td>
<td>8.38(8.38)</td>
<td>10.4(10.4)</td>
<td>44.0 (43.96)</td>
</tr>
<tr>
<td>1.50</td>
<td>4.75(4.76)</td>
<td>5.75(5.75)</td>
<td>14.97(14.51)</td>
</tr>
<tr>
<td>2.00</td>
<td>3.34(3.35)</td>
<td>4.01(4.0)</td>
<td>6.3 (6.27)</td>
</tr>
<tr>
<td>2.50</td>
<td>2.62(2.62)</td>
<td>3.11(3.1)</td>
<td>3.24(3.20)</td>
</tr>
<tr>
<td>3.00</td>
<td>2.19(2.2)</td>
<td>2.57(2.58)</td>
<td>2.0 (1.94)</td>
</tr>
<tr>
<td>4.00</td>
<td>1.71(1.7)</td>
<td>2.01</td>
<td>1.19 (1.18)</td>
</tr>
</tbody>
</table>

RESULTS AND DISCUSSION
A 1-\( \sigma \) shift would be detected in either 8.38 or 10.4 samples on cusum while Shewhart control chart requires 43.96 samples. Also, to detect the shift of order 3-\( \sigma \) or greater than 3-\( \sigma \), cusum with \( h = 4 \) or 5 requires larger samples than the Shewhart control chart. If one choose \( h = 4.77 \), this will provides a CUSUM with \( ARL_{0} = 370 \). The CUSUM is an efficient alternative to Shewhart procedures. Though CUSUM charts have not as simple to operate as Shewhart charts, they have been shown to be more efficient in detecting small shifts in the mean of a process. The analysis of ARL for CUSUM control chart shows better performance than Shewhart control charts when it is desired to detect the shifts in the mean of size 1.5 sigma or less. The simulation approach for computation of ARL shows a slight change in ARL values when the size of shift is very small. Larger shifts show almost same ARL computations.

REFERENCES